

**INTERSECTION THEORY IN
DIFFERENTIAL ALGEBRAIC GEOMETRY:
GENERIC INTERSECTIONS AND
THE DIFFERENTIAL CHOW FORM**

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ABSTRACT. In this paper, an intersection theory for generic differential polynomials is presented. The intersection of an irreducible differential variety of dimension d and order h with a generic differential hypersurface of order s is shown to be an irreducible variety of dimension $d - 1$ and order $h + s$. As a consequence, the dimension conjecture for generic differential polynomials is proved. Based on intersection theory, the Chow form for an irreducible differential variety is defined and most of the properties of the Chow form in the algebraic case are established for its differential counterpart. Furthermore, the generalized differential Chow form is defined and its properties are proved. As an application of the generalized differential Chow form, the differential resultant of $n + 1$ generic differential polynomials in n variables is defined and properties similar to that of the Macaulay resultant for multivariate polynomials are proved.

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1. INTRODUCTION

Differential algebra or differential algebraic geometry founded by Ritt and Kolchin aims to study algebraic differential equations in a similar way in which polynomial equations are studied in commutative algebra or algebraic geometry [32, 19]. Therefore, the basic concepts of commutative differential algebra are based on those of commutative algebra. An excellent survey on this subject can be found in [4].

It is known that, for many properties in algebraic geometry, their differential counterparts are much more difficult to prove and some of them are still open problems. For instance, many of the 16 questions proposed by Ritt in his classic book *Differential Algebra* [32, p.177] are still not solved. In this paper, two naturally connected problems in differential algebraic geometry are studied: the differential dimension conjecture for generic differential polynomials and the differential Chow form.

The first part of the paper is concerned with the differential dimension conjecture which is one of the problems proposed by Ritt: Let F_1, \dots, F_r be differential polynomials in $\mathcal{F}\{y_1, \dots, y_n\}$ with $r < n$, where \mathcal{F} is a differential field. If the differential variety of the system $\{F_1, \dots, F_r\}$ is nonempty, then each of its components is of dimension at least $n - r$ [32, p.178].

Ritt proved that the conjecture is correct when $r = 1$, that is, any component of a differential polynomial equation in $\mathcal{F}\{y_1, \dots, y_n\}$ is of dimension $n - 1$ [32, p.57]. The general differential dimension conjecture is still open. In [8], it is shown that the differential dimension conjecture is closely related with Jacobi's bound for the order of differential polynomial systems, which is another well-known conjecture in differential algebra.

In this paper, we consider the dimension and order for the intersection of an irreducible differential variety with generic differential hypersurfaces. A differential polynomial f is said to be generic of order s and degree m if f contains all the monomials with degree less than or equal to m in y_1, \dots, y_n and their derivatives of order up to s , and the coefficients of f are differential indeterminates. A generic differential hypersurface is the set of solutions of a generic differential polynomial. We show that for generic differential hypersurfaces, we can determine the dimension and order of their intersection with an irreducible differential variety explicitly. More precisely, we will prove

Theorem 1.1. *Let \mathcal{I} be a prime differential polynomial ideal in $\mathcal{F}\{y_1, \dots, y_n\}$ with dimension d and order h and f a generic differential polynomial with order s and degree greater than zero. If $d > 0$, then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime differential polynomial*

ideal in $\mathcal{F}\langle \mathbf{u}_f \rangle \{y_1, \dots, y_n\}$ with dimension $d - 1$ and order $h + s$, where \mathbf{u}_f is the set of coefficients of f . Also, if $d = 0$, \mathcal{I}_1 is the unit ideal in $\mathcal{F}\langle \mathbf{u}_f \rangle \{y_1, \dots, y_n\}$.

As a direct consequence of this result, we show that the dimension conjecture is valid for a system of generic differential polynomials. Furthermore, the order of the system is also given explicitly.

Another purpose for studying the intersection of an irreducible differential variety with generic differential hypersurfaces is to establish the theory of the differential Chow form, which is the concern of the second part of the paper consisting of Sections 4 to 6.

The Chow form, also known as the Cayley form or the Cayley-Chow form, is a basic concept in algebraic geometry [41, 16]. More recently, the Chow form also became a powerful tool in elimination theory. This is not surprising, since the Chow form is a resultant in a certain sense. The Chow form was used as a tool to obtain deep results in transcendental number theory by Nesterenko [27] and Philippon [29]. Brownawell made a major breakthrough in elimination theory by developing new properties of the Chow form and proving an effective version of the Nullstellensatz with optimal bounds [3]. Gel'fand et al. and Sturmfels started the sparse elimination theory, which is to study the Chow form and the resultant associated with sparse polynomials [13, 40]. Eisenbud et al. proposed a new expression for the Chow form via exterior algebra and used it to give explicit formulas in many new cases [11]. Jeronimo et al. gave a bounded probabilistic algorithm which can be used to compute the Chow form, whose complexity is polynomial in the size and the geometric degree of the input equation system [17]. Other properties of the Chow form can be found in [28, 30, 38]. Given the fact that the Chow form plays an important role in both theoretic and algorithmic aspects of algebraic geometry and has applications in many fields, it is worthwhile to develop the theory of the differential Chow form and hope that it will play a similar role as its algebraic counterpart.

Let V be an irreducible differential variety of dimension d in an n -dimensional differential affine space and

$$\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \dots + u_{in}y_n \quad (i = 0, \dots, d)$$

$d + 1$ generic hyperplanes in differential variables y_1, \dots, y_n , where u_{ij} ($i = 0, \dots, d$; $j = 0, \dots, n$) are differential indeterminates. The differential Chow form of V is roughly defined to be the elimination differential polynomial in u_{ij} for the intersection of V with $\mathbb{P}_i = 0$ ($i = 0, \dots, d$). More intuitively, the differential Chow form of V can be roughly considered as the condition on the coefficients of \mathbb{P}_i such that these $d + 1$ hyperplanes will meet V . We will show that most of the properties of the Chow form in the algebraic case presented in [16, 41] can be generalized to the differential case. Precisely, we will prove

Theorem 1.2. *Let V be an irreducible differential variety with dimension d and order h over a differential field \mathcal{F} and $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) \in \mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d\}$ the Chow form of V where $\mathbf{u}_i = (u_{i0}, u_{i1}, \dots, u_{in})$ ($i = 0, 1, \dots, d$). Then $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ has the following properties:*

1. $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ is differentially homogenous of the same degree in each set \mathbf{u}_i and $\text{ord}(F, u_{ij}) = h$ for all u_{ij} occurring in F .

2. $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ can be factored uniquely into the following form:

$$\begin{aligned} F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) &= A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) \prod_{\tau=1}^g (u_{00}^{(h)} + \sum_{\rho=1}^n u_{0\rho}^{(h)} \xi_{\tau\rho} + t_\tau) \\ &= A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) \prod_{\tau=1}^g (u_{00} + \sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})^{(h)}, \end{aligned}$$

where $g = \deg(F, u_{00}^{(h)})$, $\xi_{\tau\rho}$ are in a differential extension field \mathcal{F}_τ of \mathcal{F} , and $(u_{00} + \sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})^{(h)}$ is the h -th derivative of $(u_{00} + \sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})$. The first “=” is obtained by factoring $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ as an algebraic polynomial in the variables $u_{00}^{(h)}, u_{01}^{(h)}, \dots, u_{0n}^{(h)}$, while the second “=” is a differential expression to be explained in Section 4.4 of this paper.

3. $\Xi_\tau = (\xi_{\tau 1}, \dots, \xi_{\tau n})$ ($\tau = 1, \dots, g$) are generic points of V . Also, they are the only elements of V lying on the differential hyperplanes $\mathbb{P}_\sigma = 0$ ($\sigma = 1, \dots, d$) as well as on the algebraic hyperplanes ${}^a\mathbb{P}_0^{(l)} = 0$ ($l = 0, \dots, h - 1$).

4. Suppose that \mathbf{u}_i ($i = 0, \dots, d$) are differentially specialized to sets \mathbf{v}_i of specific elements in \mathcal{F} and $\overline{\mathbb{P}}_i$ ($i = 0, \dots, d$) are obtained by substituting \mathbf{u}_i by \mathbf{v}_i in \mathbb{P}_i . If $\overline{\mathbb{P}}_i = 0$ ($i = 0, \dots, d$) meet V , then $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$. Furthermore, if $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$ and $S_F(\mathbf{v}_0, \dots, \mathbf{v}_d) \neq 0$, then the $d + 1$ hyperplanes $\overline{\mathbb{P}}_i = 0$ ($i = 0, \dots, d$) meet V , where $S_F = \frac{\partial F}{\partial u_{00}^{(h)}}$.

In the above theorem g is called the leading differential degree of V . From the third statement of the theorem, we see that V intersects with $\mathbb{P}_\sigma = 0$ ($\sigma = 1, \dots, d$) and ${}^a\mathbb{P}_0^{(l)} = 0$ ($l = 0, \dots, h - 1$) in exactly g points. So the leading differential degree satisfies similar properties with the degree for an algebraic variety.

Furthermore, we prove that the four conditions given in Theorem 1.2 are also the sufficient conditions for a differential polynomial $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ to be the Chow form for an order un-mixed differential variety or a differential algebraic cycle. As a consequence of this result, we define the Chow quasi-variety for a class of differential algebraic cycles in the sense that each point in the Chow quasi-variety represents a differential algebraic cycle V in that class via the Chow form of V . These are clearly generalizations of the algebraic Chow variety and algebraic cycles [13, 16].

Note that both the differential Chow form and the generators of the differential Chow quasi-variety are proven to be differentially homogenous. Further developments of these concepts may need the knowledge of differential projective space [24], which has been ignored to a certain degree in the past.

In [29], Philippon considered the intersection of a variety of dimension d with $d + 1$ homogeneous polynomials with generic coefficients and developed the theory for an elimination form which can be regarded as a type of generalized Chow form. In [2], Bost, Gillet, and Soulé further generalized the concept to generalized Chow divisors of cycles and estimated their heights. In this paper, we will introduce the generalized differential Chow form which is roughly defined to be the elimination differential polynomial obtained by intersecting an irreducible differential variety V of dimension d with $d + 1$ generic differential hypersurfaces. We show that the generalized differential Chow form satisfies similar properties to that given in Theorem 1.2.

As an application of the generalized differential Chow form, we can define the differential resultant. The differential resultant for two nonlinear differential polynomials in one variable was studied by Ritt in [31, p.47]. General differential resultants were defined by Carra' Ferro [5, 6] using Macaulay's definition of an algebraic resultant of polynomials. But, the treatment in [5] is not complete. For instance, the differential resultant for two generic differential polynomials with degree two and order one in one variable is always zero if using the definition in [5]. Differential resultants for linear ordinary differential polynomials were studied by Rueda and Sendra in [36]. In this paper, a rigorous definition for the differential resultant of $n + 1$ generic differential polynomials in n variables is given as the generalized differential Chow form of the prime differential ideal $\mathcal{I} = [0]$. In this way, we obtain the following properties for differential resultants, which are similar to that of the Macaulay resultant for $n + 1$ algebraic polynomials in n variables.

Theorem 1.3. *Let $\mathbb{P}_i (i = 0, \dots, n)$ be generic differential polynomials in n differential variables y_1, \dots, y_n with orders s_i , degrees m_i , and degree zero terms u_{i0} , respectively. Let $R(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n)$ be the differential resultant of $\mathbb{P}_0, \dots, \mathbb{P}_n$, where \mathbf{u}_i is the set of coefficients of \mathbb{P}_i . Then*

- a) $R(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n)$ is differentially homogeneous in each \mathbf{u}_i and is of order $h_i = s - s_i$ in \mathbf{u}_i ($i = 0, \dots, n$) with $s = \sum_{l=0}^n s_l$.
- b) There exist $\xi_{\tau\rho} (\rho = 1, \dots, n)$ in the differential extension fields $\mathcal{F}_\tau (\tau = 1, \dots, t_0)$ of \mathcal{F} such that

$$R(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n) = A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n) \prod_{\tau=1}^{t_0} \mathbb{P}_0(\xi_{\tau 1}, \dots, \xi_{\tau n})^{(h_0)},$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n)$ is a differential polynomial in \mathbf{u}_i , $t_0 = \deg(R, u_{00}^{(h_0)})$, $\mathbb{P}_0(\xi_{\tau 1}, \dots, \xi_{\tau n})^{(h_0)}$ is the h_0 -th derivative of $\mathbb{P}_0(\xi_{\tau 1}, \dots, \xi_{\tau n})$, and $(\xi_{\tau 1}, \dots, \xi_{\tau n}) (\tau = 1, \dots, t_0)$ are certain generic points of the zero-dimensional prime differential ideal $[\mathbb{P}_1, \dots, \mathbb{P}_n]$.

c) The differential resultant can be written as a linear combination of \mathbb{P}_i and their derivatives up to the order $s - s_i$ ($i = 0, \dots, n$). Precisely, we have

$$R(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n) = \sum_{i=0}^n \sum_{j=0}^{s-s_i} h_{ij} \mathbb{P}_i^{(j)}.$$

In the above expression, $h_{ij} \in \mathcal{F}\langle \mathbf{u} \rangle [y_1, \dots, y_n, \dots, y_1^{(s)}, \dots, y_n^{(s)}]$ have degrees at most $(sn + n)^2 D^{sn+n} + D(sn + n)$, where $\mathbf{u} = \bigcup_{i=0}^n \mathbf{u}_i \setminus \{u_{00}, \dots, u_{n0}\}$, $y_i^{(j)}$ is the j -th derivative of y_i , and $D = \max\{m_0, m_1, \dots, m_n\}$.

d) Suppose that $\mathbf{u}_i (i = 0, \dots, n)$ are differentially specialized to sets \mathbf{v}_i of specific elements in \mathcal{F} and $\overline{\mathbb{P}}_i (i = 0, \dots, n)$ are obtained by substituting \mathbf{u}_i by \mathbf{v}_i in \mathbb{P}_i . If $\overline{\mathbb{P}}_i = 0 (i = 0, \dots, n)$ have a common solution, then $R(\mathbf{v}_0, \dots, \mathbf{v}_n) = 0$. On the other hand, if $R(\mathbf{v}_0, \dots, \mathbf{v}_n) = 0$ and $S_R(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$, then $\overline{\mathbb{P}}_i = 0 (i = 0, \dots, n)$ have a common solution, where $S_R = \frac{\partial R}{\partial u_{00}^{(h_0)}}$.

As a prerequisite result, we prove a useful property of differential specializations, which roughly asserts that if a set of differential polynomial functions in a set of indeterminates are differentially dependent, then they are still differentially dependent when the indeterminates are specialized to any concrete values. This property

plays a key role throughout this paper. The algebraic version of this result is also a key result in algebraic elimination theory ([15, p.168], [43, p.176]).

It is not straightforward to extend the intersection theory for generic polynomials and the theory of Chow forms from the algebraic case to the differential case. Due to the complicated structure of differential polynomials, most proofs in the algebraic case cannot be directly used in the differential case. In particular, we need to consider the orders of differential polynomials, which is not an issue in the algebraic case. For instance, the second property of the differential Chow form in Theorem 1.2 has a different form as its algebraic counterpart.

One of the main tools used in the paper is the theory of characteristic sets developed by Ritt [32, p.47]. The algorithmic character of Ritt's work on differential algebra is mainly due to the usage of characteristic sets. Properties of characteristic sets proved more recently in [1, 7, 10, 12, 43] will also be used in this paper.

The rest of this paper is organized as follows. In Section 2, we will present the notation and preliminary results used in this paper. In Section 3, the intersection theory for generic differential polynomials is given and Theorem 1.1 is proved. In Section 4, the Chow form for an irreducible differential variety is defined and its properties will be proved. Basically, we will prove Theorem 1.2. In Section 5, necessary and sufficient conditions for a differential polynomial to be the Chow form of a differential algebraic cycle is given and the Chow quasi-variety for a class of differential algebraic cycles is defined. In Section 6, we present the theory of the generalized differential Chow form and the differential resultant. Theorem 1.3 will be proved. In Section 7, we present the conclusion and propose several problems for further study.

2. PRELIMINARIES

In this section, some basic notation and preliminary results in differential algebra will be given. For more details about differential algebra, please refer to [32, 19, 25, 4, 39].

2.1. Differential polynomial algebra and Kolchin topology. Let \mathcal{F} be a fixed ordinary differential field of characteristic zero, with a derivation δ . An element $c \in \mathcal{F}$ such that $\delta(c) = 0$ is called a constant of \mathcal{F} . In this paper, unless otherwise indicated, δ is kept fixed during any discussion and we use primes and exponents (i) to indicate derivatives under δ . Let Θ denote the free commutative semigroup with unit (written multiplicatively) generated by δ . Throughout the paper, we shall often use the prefix “ δ -” as a synonym of “differential” or “differentially”.

A typical example of a differential field is $\mathbb{Q}(t)$, which is the field of rational functions in variable t with $\delta = \frac{d}{dt}$.

Let \mathcal{G} be a δ -extension field of \mathcal{F} and S a subset of \mathcal{G} . We will denote respectively by $\mathcal{F}[S]$, $\mathcal{F}(S)$, $\mathcal{F}\{S\}$, and $\mathcal{F}\langle S \rangle$ the smallest subring, the smallest subfield, the smallest δ -subring, and the smallest δ -subfield of \mathcal{G} containing \mathcal{F} and S . If we denote $\Theta(S)$ to be the smallest subset of \mathcal{G} containing S and stable under δ , we have $\mathcal{F}\{S\} = \mathcal{F}[\Theta(S)]$ and $\mathcal{F}\langle S \rangle = \mathcal{F}(\Theta(S))$. A δ -extension field \mathcal{G} of \mathcal{F} is said to be finitely generated if \mathcal{G} has a finite subset S such that $\mathcal{G} = \mathcal{F}\langle S \rangle$.

A subset Σ of a δ -extension field \mathcal{G} of \mathcal{F} is said to be δ -dependent over \mathcal{F} if the set $(\theta\alpha)_{\theta \in \Theta, \alpha \in \Sigma}$ is algebraically dependent over \mathcal{F} , and is said to be δ -independent over \mathcal{F} , or to be a family of δ -indeterminates over \mathcal{F} (abbr. δ - \mathcal{F} -indeterminates) in

the contrary case. In the case where Σ consists of one element α , we say that α is δ -algebraic or δ -transcendental over \mathcal{F} , respectively. A maximal subset Ω of \mathcal{G} which is δ -independent over \mathcal{F} is said to be a δ -transcendence basis of \mathcal{G} over \mathcal{F} . We use $\text{d.tr.deg } \mathcal{G}/\mathcal{F}$ (see [19, pp.105-109]) to denote the δ -transcendence degree of \mathcal{G} over \mathcal{F} , which is the cardinal number of Ω . Considering \mathcal{F} and \mathcal{G} as ordinary algebraic fields, we denote the algebraic transcendence degree of \mathcal{G} over \mathcal{F} by $\text{tr.deg } \mathcal{G}/\mathcal{F}$. If S is a set of δ -indeterminates over \mathcal{F} , then we call $\mathcal{F}\langle S \rangle$ a *pure δ -extension* of \mathcal{F} .

A homomorphism φ from a δ -ring (\mathcal{R}, δ) to a δ_1 -ring (\mathcal{S}, δ_1) is a *differential homomorphism* if $\varphi \circ \delta = \delta_1 \circ \varphi$. If \mathcal{R}_0 is a common δ -subring of \mathcal{R} and \mathcal{S} and if the homomorphism φ leaves every element of \mathcal{R}_0 invariant, it is said to be over \mathcal{R}_0 and is denoted by δ - \mathcal{R}_0 -homomorphism. If, in addition, \mathcal{R} is an integral domain and \mathcal{S} is a δ -field, φ is called a δ -specialization of \mathcal{R} into \mathcal{S} .

A δ -extension field \mathcal{E} of \mathcal{F} is called a *universal δ -extension field* if for any finitely generated δ -extension field \mathcal{F}_1 of \mathcal{F} in \mathcal{E} and any finitely generated δ -extension field \mathcal{F}_2 of \mathcal{F}_1 not necessarily in \mathcal{E} , \mathcal{F}_2 can be embedded in \mathcal{E} over \mathcal{F}_1 , i.e. there exists a δ -extension field of \mathcal{F}_1 in \mathcal{E} that is δ -isomorphic to \mathcal{F}_2 over \mathcal{F}_1 . Such a δ -universal extension field of \mathcal{F} always exists ([19, Theorem 2, p.134]). By definition, any finitely generated δ -extension field of \mathcal{F} can be embedded over \mathcal{F} into \mathcal{E} , and \mathcal{E} is a universal δ -extension field of every finitely generated δ -extension field of \mathcal{F} . In particular, for any natural number n , we can find in \mathcal{E} a subset of \mathcal{E} of cardinality n whose elements are δ -independent over \mathcal{F} . Throughout the present paper, \mathcal{E} stands for a fixed universal δ -extension field of \mathcal{F} .

Now suppose $\mathbb{Y} = \{y_1, y_2, \dots, y_n\}$ is a set of δ -indeterminates over \mathcal{E} . For any $y \in \mathbb{Y}$, denote $\delta^k y$ by $y^{(k)}$. The elements of $\mathcal{F}\{\mathbb{Y}\} = \mathcal{F}[y_j^{(k)} : j = 1, \dots, n; k \in \mathbb{N}]$ are called δ -polynomials over \mathcal{F} in \mathbb{Y} , and $\mathcal{F}\{\mathbb{Y}\}$ itself is called the δ -polynomial ring over \mathcal{F} in \mathbb{Y} . A δ -polynomial ideal \mathcal{I} in $\mathcal{F}\{\mathbb{Y}\}$ is an ordinary algebraic ideal which is closed under derivation, i.e. $\delta(\mathcal{I}) \subset \mathcal{I}$. In this paper, by δ -ideals we mean δ -polynomial ideals, and by a δ - \mathcal{F} -ideal we mean an ideal in $\mathcal{F}\{\mathbb{Y}\}$. Also, a prime (resp. radical) δ -ideal is a δ -ideal which is prime (resp. radical) as an ordinary algebraic polynomial ideal. For convenience, a prime δ -ideal is assumed not to be the unit ideal in this paper.

By a δ -affine space we mean any one of the sets \mathcal{E}^n ($n \in \mathbb{N}$). An element $\eta = (\eta_1, \dots, \eta_n)$ of \mathcal{E}^n will be called a point. Let Σ be a subset of δ -polynomials in $\mathcal{F}\{\mathbb{Y}\}$. A point $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{E}^n$ is called a δ -zero of Σ if $f(\eta) = 0$ for any $f \in \Sigma$. The set of δ -zeros of Σ is denoted by $\mathbb{V}(\Sigma)$, which is called a δ -variety defined over \mathcal{F} (abbr. δ - \mathcal{F} -variety). Also, for any $D \in \mathcal{F}\{\mathbb{Y}\}$, $\mathbb{V}(\Sigma/D) = \mathbb{V}(\Sigma) \setminus \mathbb{V}(D)$ is called a δ -quasi-variety. For convenience, we also call $\bigcup_{i=1}^m \mathbb{V}(\Sigma_i/D_i)$ a δ -quasi-variety, where Σ_i and D_i are δ -polynomial sets and δ -polynomials, respectively. The δ -varieties in \mathcal{E}^n (resp. the δ - \mathcal{F} -varieties in \mathcal{E}^n) are the closed sets in a topology called the *Kolchin topology* (resp. the *Kolchin \mathcal{F} -topology*).

For a δ - \mathcal{F} -variety V , we denote $\mathbb{I}(V)$ to be the set of all δ -polynomials in $\mathcal{F}\{\mathbb{Y}\}$ that vanish at every point of V . Clearly, $\mathbb{I}(V)$ is a radical δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$, and there exists a bijective correspondence between Kolchin \mathcal{F} -closed sets and radical δ -ideals in $\mathcal{F}\{\mathbb{Y}\}$. That is, for any δ - \mathcal{F} variety V , $\mathbb{V}(\mathbb{I}(V)) = V$, and for any radical δ -ideal \mathcal{I} in $\mathcal{F}\{\mathbb{Y}\}$, $\mathbb{I}(\mathbb{V}(\mathcal{I})) = \mathcal{I}$.

Similarly as in algebraic geometry, an \mathcal{F} -irreducible δ -variety can be defined, and there is a bijective correspondence between \mathcal{F} -irreducible δ -varieties and prime δ -ideals in $\mathcal{F}\{\mathbb{Y}\}$. A point $\eta \in \mathbb{V}(\mathcal{I})$ is called a *generic point* of a prime δ -ideal

$\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$ if for any δ -polynomial $p \in \mathcal{F}\{\mathbb{Y}\}$ we have $p(\eta) = 0 \Leftrightarrow p \in \mathcal{I}$. It is well known that [32, p.27]

Lemma 2.1. *A non-unit δ -ideal is prime if and only if it has a generic point.*

By the definition of universal δ -fields, a prime δ -ideal over any finitely generated δ -extension field of \mathcal{F} has a generic point in \mathcal{E}^n . A point $\eta \in V$ is called a *generic point* of an irreducible δ -variety V if η is a generic point for $\mathbb{I}(V)$.

Notice that irreducibility depends on the base field over which the δ -polynomials are defined. For instance, $(\delta y_1)^2 - t$ is an irreducible δ -polynomial in $\mathbb{Q}(t)\{y_1\}$ where $\delta = \frac{\partial}{\partial t}$, but it can be factored in $\mathbb{Q}(\sqrt{t})\{y_1\}$. Thus, to emphasize the base field, in the rest of the paper we will use $\mathcal{I} = [\Sigma] \subset \mathcal{G}\{\mathbb{Y}\}$ to denote the δ -ideal generated by Σ in $\mathcal{G}\{\mathbb{Y}\}$ and use $\mathcal{G}_1 \cdot \mathcal{I}$ to denote the δ -ideal generated by \mathcal{I} in $\mathcal{G}_1\{\mathbb{Y}\}$ where $\mathcal{G}_1 \subset \mathcal{E}$ is a δ -extension field of \mathcal{G} .

2.2. Characteristic sets of a differential polynomial set. Let f be a δ -polynomial in $\mathcal{F}\{\mathbb{Y}\}$. We define the order of f w.r.t. y_i to be the greatest number k such that $y_i^{(k)}$ appears effectively in f , which is denoted by $\text{ord}(f, y_i)$. If y_i does not appear in f , then we set $\text{ord}(f, y_i) = -\infty$. The *order* of f is defined to be $\max_i \text{ord}(f, y_i)$, that is, $\text{ord}(f) = \max_i \text{ord}(f, y_i)$.

A *ranking* \mathcal{R} is a total order over $\Theta(\mathbb{Y})$, which is compatible with the derivations over the alphabet:

- 1) $\delta\theta y_j > \theta y_j$ for all derivatives $\theta y_j \in \Theta(\mathbb{Y})$.
- 2) $\theta_1 y_i > \theta_2 y_j \implies \delta\theta_1 y_i > \delta\theta_2 y_j$ for $\theta_1 y_i, \theta_2 y_j \in \Theta(\mathbb{Y})$.

By convention, $1 < \theta y_j$ for all $\theta y_j \in \Theta(\mathbb{Y})$.

Two important kinds of rankings are the following:

- 1) *Elimination ranking:* $y_i > y_j \implies \delta^k y_i > \delta^l y_j$ for any $k, l \geq 0$.
- 2) *Orderly ranking:* $k > l \implies \delta^k y_i > \delta^l y_j$, for any $i, j \in \{1, 2, \dots, n\}$.

Let p be a δ -polynomial in $\mathcal{F}\{\mathbb{Y}\}$ and \mathcal{R} a ranking endowed on it. The greatest derivative w.r.t. \mathcal{R} which appears effectively in p is called the *leader* of p , which will be denoted by u_p or $\text{ld}(p)$. The two conditions mentioned above imply that the leader of θp is θu_p for $\theta \in \Theta$. Let the degree of p in u_p be d . We rewrite p as an algebraic polynomial in u_p . Then

$$p = I_d u_p^d + I_{d-1} u_p^{d-1} + \dots + I_0.$$

We call I_d the *initial* of p and denote it by I_p . The partial derivative of p w.r.t. u_p is called the *separant* of p , which will be denoted by S_p . Clearly, S_p is the initial of any proper derivative of p . The *rank* of p is u_p^d , and we denote it by $\text{rk}(p)$. For any two δ -polynomials p, q in $\mathcal{F}\{\mathbb{Y}\} \setminus \mathcal{F}$, p is said to be of *lower rank* than q if either $u_p < u_q$ or $u_p = u_q = u$ and $\deg(p, u) < \deg(q, u)$. By convention, any element of \mathcal{F} is of lower rank than elements of $\mathcal{F}\{\mathbb{Y}\} \setminus \mathcal{F}$. We denote $p \preceq q$ if and only if either p is of lower rank than q or they have the same rank. Clearly, \preceq is a total ordering of $\mathcal{F}\{\mathbb{Y}\}$.

Let p and q be two δ -polynomials and u_p^d the rank of p . q is said to be *partially reduced* w.r.t. p if no proper derivatives of u_p appear in q . q is said to be *reduced* w.r.t. p if q is partially reduced w.r.t. p and $\deg(q, u_p) < d$. Let \mathcal{A} be a set of δ -polynomials. \mathcal{A} is said to be an *auto-reduced set* if each δ -polynomial of \mathcal{A} is reduced w.r.t. any other element of \mathcal{A} . Every auto-reduced set is finite.

Let \mathcal{A} be an auto-reduced set. We denote $H_{\mathcal{A}}$ to be the set of all the initials and separants of \mathcal{A} and $H_{\mathcal{A}}^{\infty}$ to be the minimal multiplicative set containing $H_{\mathcal{A}}$. The

saturation ideal of \mathcal{A} is defined to be

$$\text{sat}(\mathcal{A}) = [\mathcal{A}] : H_{\mathcal{A}}^{\infty} = \{p \in \mathcal{F}\{\mathbb{Y}\} \mid \exists h \in H_{\mathcal{A}}^{\infty}, \text{ s. t. } hp \in [\mathcal{A}]\}.$$

Let $\mathcal{A} = A_1, A_2, \dots, A_s$ and $\mathcal{B} = B_1, B_2, \dots, B_l$ be two auto-reduced sets with the A_i, B_j arranged in nondecreasing ordering. \mathcal{A} is said to be of *lower rank* than \mathcal{B} if either 1) there is some $k (\leq \min\{s, l\})$ such that for each $i < k$, A_i has the same rank as B_i , and $A_k \prec B_k$ or 2) $s > l$, and for each $i \in \{1, 2, \dots, l\}$, A_i has the same rank as B_i . It is easy to see that the above definition really introduces a partial ordering among all auto-reduced sets. Any sequence of auto-reduced sets steadily decreasing in ordering $\mathcal{A}_1 \succ \mathcal{A}_2 \succ \dots \mathcal{A}_k \succ \dots$ is necessarily finite.

Let $\mathcal{A} = A_1, A_2, \dots, A_t$ be an auto-reduced set with S_i and I_i as the separant and initial of A_i , and f any δ -polynomial. Then there exists an algorithm, called Ritt's algorithm of reduction, which reduces f w.r.t. \mathcal{A} to a δ -polynomial r that is reduced w.r.t. \mathcal{A} , satisfying the relation

$$\prod_{i=1}^t S_i^{d_i} I_i^{e_i} \cdot f \equiv r, \text{ mod } [\mathcal{A}],$$

for nonnegative integers $d_i, e_i (i = 1, 2, \dots, t)$. We call r the *pseudo remainder* of f w.r.t. \mathcal{A} .

An auto-reduced set \mathcal{C} contained in a δ -polynomial set \mathcal{S} is said to be a *characteristic set* of \mathcal{S} if \mathcal{S} does not contain any nonzero element reduced w.r.t. \mathcal{C} . All the characteristic sets of \mathcal{S} have the same and minimal rank among all auto-reduced sets contained in \mathcal{S} . A characteristic set \mathcal{C} of a δ -ideal \mathcal{J} reduces to zero all elements of \mathcal{J} . If the δ -ideal is prime, \mathcal{C} reduces to zero only the elements of \mathcal{J} and we have $\mathcal{J} = \text{sat}(\mathcal{C})$ ([19, Lemma 2, p.167]).

In polynomial algebra, let $\mathcal{A} = A_1, A_2, \dots, A_t$ be an algebraic auto-reduced set arranged in nondecreasing order. \mathcal{A} is called an *irreducible auto-reduced set* if for any $1 \leq i \leq t$, there cannot exist any relation of the form

$$T_i A_i = B_i C_i, \text{ mod } (A_1, \dots, A_{i-1})$$

where B_i, C_i are polynomials with the same leader as A_i , T_i is a polynomial with a lower leader than A_i , and B_i, C_i, T_i are reduced w.r.t. A_1, \dots, A_{i-1} ([42]). Equivalently, an algebraic auto-reduced set \mathcal{A} is irreducible if and only if there exist no polynomials P and Q which are reduced w.r.t. \mathcal{A} and $PQ \in \text{asat}(\mathcal{A}) = (\mathcal{A}) : I_{\mathcal{A}}^{\infty}$, where $I_{\mathcal{A}}^{\infty}$ stands for the set of all products of powers of I_{A_i} .

In ordinary differential algebra, we can define an auto-reduced set to be *irreducible* if, when considered as an algebraic auto-reduced set in the underlying polynomial ring, it is irreducible. We have ([32, p.107]).

Theorem 2.2. *Let \mathcal{A} be an auto-reduced set. Then a necessary and sufficient condition for \mathcal{A} to be a characteristic set of a prime δ -ideal is that \mathcal{A} is irreducible. Moreover, in the case \mathcal{A} is irreducible, $\text{sat}(\mathcal{A}) = [\mathcal{A}] : H_{\mathcal{A}}^{\infty}$ is prime with \mathcal{A} being a characteristic set of it.*

Remark 2.3. A set of δ -polynomials $\mathcal{A} = \{A_1, \dots, A_p\}$ is called a δ -chain if the following conditions are satisfied:

- 1) the leaders of A_i are δ -auto-reduced,
- 2) each A_i is partially reduced w.r.t. all the others,
- 3) no initial of an element of \mathcal{A} is reduced to zero by \mathcal{A} .

Similar properties to auto-reduced sets can be developed for δ -chains [10]. In particular, we can define a δ -characteristic set of a δ -ideal \mathcal{I} to be a δ -chain contained in \mathcal{I} of minimal rank among all the δ -chains contained in \mathcal{I} . So, in this paper we will not distinguish auto-reduced sets and δ -chains. Note that we can also use the weak δ -chains introduced in [7].

2.3. Dimension and order of a prime differential polynomial ideal. Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ and $\xi = (\xi_1, \dots, \xi_n)$ a generic point of \mathcal{I} [19, p.19]. The *dimension* of \mathcal{I} or $\mathbb{V}(\mathcal{I})$ is defined to be the δ -transcendence degree of $\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle$ over \mathcal{F} , that is, $\dim(\mathcal{I}) = \text{d.tr.deg } \mathcal{F}\langle \xi_1, \dots, \xi_n \rangle / \mathcal{F}$.

In [32], Ritt gave another definition of the dimension of \mathcal{I} . An independent set modulo \mathcal{I} is defined to be a variable set $\mathbb{U} \subset \mathbb{Y}$ such that $\mathcal{I} \cap \mathcal{F}\{\mathbb{U}\} = \{0\}$, and in this case \mathbb{U} is also said to be δ -independent modulo \mathcal{I} . A *parametric set* of \mathcal{I} is a maximal independent set modulo \mathcal{I} . Then Ritt defined the dimension of \mathcal{I} to be the cardinal number of its parametric set. Clearly, the two definitions are equivalent.

Definition 2.4 ([22]). Let \mathcal{I} be a prime δ -ideal of $\mathcal{F}\{\mathbb{Y}\}$ with a generic point $\eta = (\eta_1, \dots, \eta_n)$. Then there exists a unique numerical polynomial $\omega_{\mathcal{I}}(t)$ such that $\omega_{\mathcal{I}}(t) = \omega_{\eta/\mathcal{F}}(t) = \text{tr.deg } \mathcal{F}\langle \eta_i^{(j)} : i = 1, \dots, n; j \leq t \rangle / \mathcal{F}$ for all sufficiently large $t \in \mathbb{N}$. $\omega_{\mathcal{I}}(t)$ is called the *δ -dimension polynomial* of \mathcal{I} .

We now define the order of a prime δ -ideal \mathcal{I} , which is also related with the characteristic set of \mathcal{I} .

Definition 2.5. For an auto-reduced set $\mathcal{A} = A_1, A_2, \dots, A_t$, with $\text{ld}(A_i) = y_{c_i}^{(o_i)}$, the order of \mathcal{A} is defined to be $\text{ord}(\mathcal{A}) = \sum_{i=1}^t o_i$, and the set $\mathbb{Y} \setminus \{y_{c_1}, \dots, y_{c_t}\}$ is called a parametric set of \mathcal{A} .

Theorem 2.6 ([34, Theorem 13]). *Let \mathcal{I} be a prime δ -ideal of dimension d . Then the δ -dimension polynomial has the form $\omega_{\mathcal{I}}(t) = d(t + 1) + h$, where h is defined to be the order of \mathcal{I} or $\mathbb{V}(\mathcal{I})$, that is, $\text{ord}(\mathcal{I}) = h$. Let \mathcal{A} be a characteristic set of \mathcal{I} under any orderly ranking. Then, $\text{ord}(\mathcal{I}) = \text{ord}(\mathcal{A})$.*

In [32], Ritt introduced the concept of relative order for a prime δ -ideal w.r.t. a particular parametric set.

Definition 2.7. Let \mathcal{I} be a prime δ -ideal of $\mathcal{F}\{\mathbb{Y}\}$, \mathcal{A} a characteristic set of \mathcal{I} w.r.t. any elimination ranking, and $\{u_1, \dots, u_d\} \subset \mathbb{Y}$ the parametric set of \mathcal{A} . The *relative order* of \mathcal{I} w.r.t. $\{u_1, \dots, u_d\}$, denoted by $\text{ord}_{u_1, \dots, u_d} \mathcal{I}$, is defined to be $\text{ord}(\mathcal{A})$.

The relative order of a prime δ -ideal \mathcal{I} can be computed from its generic points as shown by the following result ([21]).

Corollary 2.8 ([21]). *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ with a generic point (ξ_1, \dots, ξ_n) . If $\{y_1, \dots, y_d\}$ is a parametric set of \mathcal{I} , then*

$$\text{ord}_{y_1, \dots, y_d}(\mathcal{I}) = \text{tr.deg } \mathcal{F}\langle \xi_1, \dots, \xi_d \rangle \langle \xi_{d+1}, \dots, \xi_n \rangle / \mathcal{F}\langle \xi_1, \dots, \xi_d \rangle.$$

Ritt’s definition of relative order is based on the elimination ranking. Hubert proved that all characteristic sets of \mathcal{I} admitting the same parametric set have the same order [10].

Theorem 2.9 ([10]). *Let \mathcal{A} be a characteristic set of a prime δ -ideal \mathcal{I} in $\mathcal{F}\{\mathbb{Y}\}$ endowed with any ranking. The parametric set \mathbb{U} of \mathcal{A} is a maximal independent*

set modulo \mathcal{I} . Its cardinality gives the dimension of \mathcal{I} . Furthermore, the order of \mathcal{I} relative to \mathbb{U} is the order of \mathcal{A} .

Corollary 2.10. *Let \mathcal{I} be a prime δ -ideal with dimension zero and \mathcal{A} a characteristic set of \mathcal{I} w.r.t. any ranking \mathcal{R} . Then $\text{ord}(\mathcal{I}) = \text{ord}(\mathcal{A})$.*

The following result gives the relation between the order and relative order for a prime δ -ideal.

Theorem 2.11. *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$. Then $\text{ord}(\mathcal{I})$ is the maximum of all the relative orders of \mathcal{I} , that is, $\text{ord}(\mathcal{I}) = \max_{\mathbb{U}} \text{ord}_{\mathbb{U}}(\mathcal{I})$, where \mathbb{U} is any parametric set of \mathcal{I} .*

Proof. Let \mathcal{C} be a characteristic set of \mathcal{I} w.r.t. some orderly ranking. First, we claim that any relative order of \mathcal{I} is less than or equal to $\text{ord}(\mathcal{C})$. Let $\mathbb{U} = \{u_1, \dots, u_q\}$ be any parametric set of \mathcal{I} , $\{y_1, \dots, y_p\}$ ($p + q = n$) the set of the remaining variables, and \mathcal{B} any characteristic set of \mathcal{I} w.r.t. the elimination ranking $u_1 \prec \dots \prec u_q \prec y_1 \prec \dots \prec y_p$. By Theorem 2.9, it suffices to prove $\text{ord}_{\mathbb{U}}(\mathcal{I}) \leq \text{ord}(\mathcal{C})$.

Let $\eta = (\overline{u}_1, \dots, \overline{u}_q, \overline{y}_1, \dots, \overline{y}_p)$ be a generic point of \mathcal{I} . Then for sufficiently large t , the δ -dimension polynomial of \mathcal{I} is

$$\begin{aligned} &\omega_{\mathcal{I}}(t) \\ &= \omega_{\eta/\mathcal{F}}(t) \\ &= \text{tr.deg } \mathcal{F}(\delta^s \overline{u}_i, \delta^k \overline{y}_j : s, k \leq t; i = 1, \dots, q; j = 1, \dots, p) / \mathcal{F} \\ &= \text{tr.deg } \mathcal{F}(\delta^s \overline{u}_i : s \leq t) (\delta^k \overline{y}_j : k \leq t) / \mathcal{F}(\delta^s \overline{u}_i : s \leq t) + \text{tr.deg } \mathcal{F}(\delta^s \overline{u}_i : s \leq t) / \mathcal{F} \\ &= q(t + 1) + \text{tr.deg } \mathcal{F}(\delta^s \overline{u}_i : s \leq t) (\delta^k \overline{y}_j : k \leq t) / \mathcal{F}(\delta^s \overline{u}_i : s \leq t). \end{aligned}$$

Since $\omega_{\mathcal{I}}(t) = q(t + 1) + \text{ord}(\mathcal{C})$, $\text{tr.deg } \mathcal{F}(\delta^s \overline{u}_i : s \leq t) (\delta^k \overline{y}_j : k \leq t) / \mathcal{F}(\delta^s \overline{u}_i : s \leq t) = \text{ord}(\mathcal{C})$. By Corollary 2.8, we have

$$\begin{aligned} &\text{ord}_{\mathbb{U}}(\mathcal{I}) \\ &= \text{tr.deg } \mathcal{F}(\overline{u}_1, \dots, \overline{u}_q) (\delta^k \overline{y}_l : k \geq 0) / \mathcal{F}(\overline{u}_1, \dots, \overline{u}_q) \\ &= \text{tr.deg } \mathcal{F}(\overline{u}_1, \dots, \overline{u}_q) (\delta^k \overline{y}_l : k \leq t) / \mathcal{F}(\overline{u}_1, \dots, \overline{u}_q) \quad (\text{for } t \geq \text{ord}(\mathcal{B})) \\ &\leq \text{tr.deg } \mathcal{F}(\delta^s \overline{u}_i : s \leq t) (\delta^k \overline{y}_j : k \leq t) / \mathcal{F}(\delta^s \overline{u}_i : s \leq t) \\ &= \text{ord}(\mathcal{C}). \end{aligned}$$

Thus, the claim is proved.

Now, let \mathbb{U}^* be the parametric set of \mathcal{C} . Then, by Theorem 2.9, $\text{ord}(\mathcal{I}) = \text{ord}(\mathcal{C}) = \text{ord}_{\mathbb{U}^*}(\mathcal{I})$. That is, for any parametric set \mathbb{U} of \mathcal{I} , we have $\text{ord}_{\mathbb{U}}(\mathcal{I}) \leq \text{ord}(\mathcal{I})$ and there exists one parametric set \mathbb{U}^* of \mathcal{I} such that $\text{ord}_{\mathbb{U}^*}(\mathcal{I}) = \text{ord}(\mathcal{I})$. As a consequence, $\text{ord}(\mathcal{I}) = \max_{\mathbb{U}} \text{ord}_{\mathbb{U}}(\mathcal{I})$. \square

The following well-known result about the adjoining indeterminates to the base field will be used in this paper [19, 32].

Lemma 2.12. *Let $\mathbb{U} = \{u_1, \dots, u_r\} \subset \mathcal{E}$ be a set of δ - \mathcal{F} -indeterminates, \mathcal{I}_0 a prime δ -ideal of dimension d and order h in $\mathcal{F}\{\mathbb{Y}\}$, and $\mathcal{I} = \mathcal{F}\langle \mathbb{U} \rangle \cdot \mathcal{I}_0$ the δ -ideal generated by \mathcal{I}_0 in $\mathcal{F}\langle \mathbb{U} \rangle\{\mathbb{Y}\}$. Then \mathcal{I} is a prime δ -ideal in $\mathcal{F}\langle \mathbb{U} \rangle\{\mathbb{Y}\}$ of dimension d and order h and $\mathcal{I} \cap \mathcal{F}\{\mathbb{Y}\} = \mathcal{I}_0$.*

2.4. A property on differential specialization. The following lemma is a key result in algebraic elimination theory, which is used to develop the theory of Chow form ([15, pp.168-169], [43, p.176]). The result is originally given for homogenous polynomials. We will show that it also holds for nonhomogenous polynomials.

Lemma 2.13. *Let $P_i \in \mathcal{F}[\mathbb{U}, \mathbb{Y}]$ ($i = 1, \dots, m$) be polynomials in the independent indeterminates $\mathbb{U} = (u_1, \dots, u_r)$ and $\mathbb{Y} = (y_1, \dots, y_n)$. Let $\overline{\mathbb{Y}} = (\overline{y}_1, \dots, \overline{y}_n)$, where \overline{y}_i are elements of some extension field of \mathcal{F} free from $\mathcal{F}(\mathbb{U})$.¹ If $P_i(\mathbb{U}, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are algebraically dependent over $\mathcal{F}(\mathbb{U})$, then for any specialization \mathbb{U} to $\overline{\mathbb{U}} = (\overline{u}_1, \dots, \overline{u}_r) \in \mathcal{F}^r$, $P_i(\overline{\mathbb{U}}, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are algebraically dependent over \mathcal{F} .*

Proof. We sketch the proof for the case $r = 1$. Since $P_i(u_1, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are algebraically dependent over $\mathcal{F}(u_1)$, there exists some nonzero polynomial $f \in \mathcal{F}(u_1)[z_1, \dots, z_m]$ which vanishes for $z_i = P_i(u_1, \overline{\mathbb{Y}})$. By clearing fractions when necessary, we suppose $f \in \mathcal{F}[u_1, z_1, \dots, z_m]$. Now specialize u_1 to \overline{u}_1 in f ; then we have $f(\overline{u}_1; P_1(\overline{u}_1, \overline{\mathbb{Y}}), \dots, P_m(\overline{u}_1, \overline{\mathbb{Y}})) = 0$. If $f(\overline{u}_1; z_1, \dots, z_m) \neq 0$, it follows that $P_i(\overline{u}_1, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are algebraically dependent over \mathcal{F} . If $f(\overline{u}_1; z_1, \dots, z_m) = 0$, then $f(u_1; z_1, \dots, z_m) = (u_1 - \overline{u}_1)^l f_1$, where $f_1(\overline{u}_1; z_1, \dots, z_m) \neq 0$. Since $f(\overline{u}_1; P_1(\overline{u}_1, \overline{\mathbb{Y}}), \dots, P_m(\overline{u}_1, \overline{\mathbb{Y}})) = 0$, we have $f_1(\overline{u}_1; P_1(\overline{u}_1, \overline{\mathbb{Y}}), \dots, P_m(\overline{u}_1, \overline{\mathbb{Y}})) = 0$. Thus, it follows that $P_i(\overline{\mathbb{U}}, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are algebraically dependent over \mathcal{F} . \square

To generalize the above result to the differential case, we need the following lemma [32, p.35].

Lemma 2.14. *Suppose \mathcal{F} contains at least one nonconstant element. If $G \in \mathcal{F}\{u\}$ is a nonzero δ -polynomial with order r , then for any nonconstant $\eta \in \mathcal{F}$, there exists an element $c_0 + c_1\eta + c_2\eta^2 + \dots + c_r\eta^r$ which does not annul G , where c_0, \dots, c_r are constants in \mathcal{F} .*

By arbitrary constants a_1, \dots, a_s over \mathcal{F} , we mean a_1, \dots, a_s are constants in \mathcal{E} , which are algebraically independent over \mathcal{F} . As a consequence of Lemma 2.14, we have

Corollary 2.15. *Suppose \mathcal{F} contains at least one nonconstant element. If $G \in \mathcal{F}\{u\}$ is a nonzero δ -polynomial with order r , then for any nonconstant $\eta \in \mathcal{F}$ and arbitrary constants a_0, \dots, a_r over \mathcal{F} , $a_0 + a_1\eta + a_2\eta^2 + \dots + a_r\eta^r$ does not annul G .*

Proof. Suppose the contrary. Let a_0, \dots, a_r be arbitrary constants over \mathcal{F} in \mathcal{E} such that $G(a_0 + a_1\eta + \dots + a_r\eta^r) = 0$, where η is a nonconstant in \mathcal{F} . Since a_0, \dots, a_r are constants in \mathcal{E} which are algebraically independent over \mathcal{F} , $g(a_0, \dots, a_r) = G(a_0 + a_1\eta + \dots + a_r\eta^r)$ is a polynomial in $\mathcal{F}[a_0, \dots, a_r]$. Now by the hypothesis, g is a zero polynomial. Thus, for any constants $\overline{a}_i \in \mathcal{F}$ ($i = 0, \dots, r$), $g(\overline{a}_0, \dots, \overline{a}_r) = G(\overline{a}_0 + \overline{a}_1\eta + \dots + \overline{a}_r\eta^r) = 0$, which contradicts Lemma 2.14. \square

Now we prove the following result, which is crucial throughout the paper.

Theorem 2.16. *Let $\{u_1, \dots, u_r\} \subset \mathcal{E}$ be a set of δ - \mathcal{F} -indeterminates, and $P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\{\mathbb{U}, \mathbb{Y}\}$ ($i = 1, \dots, m$) δ -polynomials in the δ - \mathcal{F} -indeterminates $\mathbb{U} = (u_1, \dots, u_r)$ and $\mathbb{Y} = (y_1, \dots, y_n)$. Let $\overline{\mathbb{Y}} = (\overline{y}_1, \overline{y}_2, \dots, \overline{y}_n)$, where $\overline{y}_i \in \mathcal{E}$ are δ -free from $\mathcal{F}\langle \mathbb{U} \rangle$.*

¹By saying $\overline{\mathbb{Y}}$ is free from $\mathcal{F}(\mathbb{U})$ (resp. δ -free from $\mathcal{F}\langle \mathbb{U} \rangle$), we mean that \mathbb{U} is a set of indeterminates over $\mathcal{F}(\overline{\mathbb{Y}})$ (resp. δ -indeterminates over $\mathcal{F}\langle \overline{\mathbb{Y}} \rangle$).

If $P_i(\mathbb{U}, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are δ -dependent over $\mathcal{F}\langle \mathbb{U} \rangle$, then for any specialization \mathbb{U} to $\overline{\mathbb{U}}$ in \mathcal{F} , $P_i(\overline{\mathbb{U}}, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are δ - \mathcal{F} -dependent.

Proof. It suffices to prove the case $r = 1$. Denote u_1 by u . First, we suppose \mathcal{F} contains at least one nonconstant element.

Since $P_i(u, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are δ -dependent over $\mathcal{F}\langle u \rangle$, there exists a nonzero $G(z_1, \dots, z_m) \in \mathcal{F}\langle u \rangle\{z_1, \dots, z_m\}$ such that $G(P_1(u, \overline{\mathbb{Y}}), \dots, P_m(u, \overline{\mathbb{Y}})) = 0$. We can take $G \in \mathcal{F}\{u, z_1, \dots, z_m\}$ by clearing denominators when necessary.

Since $G(u, z_1, \dots, z_m) \neq 0$, by Corollary 2.15, for any nonconstant $\eta \in \mathcal{F}$ and arbitrary constants c_0, \dots, c_s ($s = \text{ord}(G, u)$) over $\mathcal{F}\langle \overline{\mathbb{Y}} \rangle$, $G(u^*, z_1, \dots, z_m) \neq 0$ where $u^* = \sum_{i=0}^s c_i \eta^i$. Since $G(u, P_1(u, \overline{\mathbb{Y}}), \dots, P_m(u, \overline{\mathbb{Y}})) = 0$ and u is a δ -indeterminate over $\mathcal{F}\langle \overline{\mathbb{Y}} \rangle$, when u is specialized to u^* , we have $G(u^*, P_1(u^*, \overline{\mathbb{Y}}), \dots, P_m(u^*, \overline{\mathbb{Y}})) = 0$. Regarding G as an algebraic polynomial over \mathcal{F} in c_i ($i = 0, \dots, s$) and $z_i^{(j)}$ ($i = 1, \dots, m; j \geq 0$) which appear effectively, we have

$$G(c_0, \dots, c_s, \dots, z_i^{(j)}, \dots) \neq 0,$$

while

$$G(c_0, \dots, c_s, \dots, (P_i(u^*, \overline{\mathbb{Y}}))^{(j)}, \dots) = 0.$$

So $P_i(u^*, \overline{\mathbb{Y}})^{(j)}$ ($i = 1, \dots, m; j \geq 0$) are algebraically dependent over $\mathcal{F}(c_0, \dots, c_s)$, by Lemma 2.13, when the c_i are specialized to constants \bar{c}_i in \mathcal{F} , the corresponding $P_i(\bar{u}^*, \overline{\mathbb{Y}})^{(j)}$ ($i = 1, \dots, m$) are algebraically dependent over \mathcal{F} , where $\bar{u}^* = \sum_{i=0}^s \bar{c}_i \eta^i$. That is, $P_i(\bar{u}^*, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are δ -dependent over \mathcal{F} .

To complete the proof, if \bar{u} is a nonconstant, as above we take $\eta = \bar{u}$, and specialize $c_1 \rightarrow 1$ and other c_i to zero. Otherwise we take η as an arbitrary nonconstant and specialize $c_0 \rightarrow \bar{u}$ and other c_i to zero. Then in either case, u is specialized to \bar{u} , and we have completed the proof in the case where \mathcal{F} contains at least one nonconstant element.

If \mathcal{F} consists of constant elements, take $v \in \mathcal{E}$ to be a δ -indeterminate over $\mathcal{F}\langle \mathbb{U}, \overline{\mathbb{Y}} \rangle$. Now we consider in the δ -field $\mathcal{F}\langle v \rangle$. Following the first case, for any specialization \mathbb{U} to $\overline{\mathbb{U}} \subset \mathcal{F}$, we can show that $P_i(\overline{\mathbb{U}}, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are δ -dependent over $\mathcal{F}\langle v \rangle$. Since v is a δ -indeterminate over $\mathcal{F}\langle \overline{\mathbb{Y}} \rangle$, $P_i(\overline{\mathbb{U}}, \overline{\mathbb{Y}})$ ($i = 1, \dots, m$) are δ -dependent over \mathcal{F} . □

From the proof above, we can obtain the following result easily:

Corollary 2.17. *Let $\{u_1, \dots, u_r\} \subset \mathcal{E}$ be a set of δ - \mathcal{F} -indeterminates, and $P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\{\mathbb{U}, \mathbb{Y}\}$ ($i = 1, \dots, m$) δ -polynomials in the δ -indeterminates $\mathbb{U} = (u_1, \dots, u_r)$ and $\mathbb{Y} = (y_1, \dots, y_n)$. Let $\overline{\mathbb{Y}} = (\overline{y}_1, \overline{y}_2, \dots, \overline{y}_n)$, where $\overline{y}_i \in \mathcal{E}$ are free from $\mathcal{F}\langle \mathbb{U} \rangle$. If the set $(P_i(\mathbb{U}, \overline{\mathbb{Y}}))^{(\sigma_{ij})}$ ($i = 1, \dots, m; j = 1, \dots, n_i$) are algebraically dependent over $\mathcal{F}\langle \mathbb{U} \rangle$, then for any specialization \mathbb{U} to $\overline{\mathbb{U}}$ in \mathcal{F} , $(P_i(\overline{\mathbb{U}}, \overline{\mathbb{Y}}))^{(\sigma_{ij})}$ ($i = 1, \dots, m; j = 1, \dots, n_i$) are algebraically dependent over \mathcal{F} .*

Now we give an example to illustrate the proof of Theorem 2.16.

Example 2.18. In this example $r = m = n = 1$ and $\mathcal{F} = \mathbb{Q}(t)$ with $\delta = \frac{d}{dt}$. Let $P(u, y) = uy$. Suppose that ξ is a generic point of $\mathcal{I} = \text{sat}(y^2 - 2y) \subset \mathbb{Q}(t)\{y\}$ that is δ -free from $\mathbb{Q}(t)\langle u \rangle$. Let $G(u; z) = u'^2 z''^2 + 2u(u'^2 - uu'')z'' - 2u'u''z'z'' + u''^2 z'^2 + z'(-8u'^3 + 6uu'u'') - 3u^2 u'^2 + 2u^3 u'' \in \mathbb{Q}(t)\{u, z\}$. Let $\overline{y} = \xi$. It is easy to verify that $G(u; z) \neq 0$ and $G(u; P(u, \overline{y})) = 0$. That is, $P(u, \overline{y})$ is δ -algebraic over $\mathbb{Q}(t)\langle u \rangle$. We now proceed to show that when u is specialized to $\bar{u} = 1$, $P(\bar{u}, \overline{y})$ is δ -algebraic over $\mathbb{Q}(t)$.

Clearly, $G(\bar{u}; z) = 0$. In the algebraic case, following the proof of Lemma 2.13, $u - \bar{u}$ must be a factor of $G(u; z)$. Removing all factors of the form $u - \bar{u}$, we obtain a new δ -polynomial $G_1(u; z)$ which still satisfies $G_1(u; P(u, \bar{y})) = 0$ and $G_1(\bar{u}; z) \neq 0$. But, in the differential case, $G(u; z)$ may not contain any factor involving $u - \bar{u}$ and its derivatives.

Follow the steps in Theorem 2.16, and let c_0, c_1, c_2 be arbitrary constants over $\mathbb{Q}(t)\langle \xi \rangle$ and $\eta = t$. Denote $u^* = c_0 + c_1t + c_2t^2$. Then $G(u^*; z) = -40z'c_2^3t^3 - 6c_0c_1^3t - 12c_0c_2^3t^4 - 14c_1^3t^3c_2 - 27c_1^2t^4c_2^2 + 4z'^2c_2^2 - 8z'c_1^3 - 3c_0^2c_1^2 - 3c_1^4t^2 - 8c_2^4t^6 + 4c_0^3c_2 + z''^2c_1^2 - 36z'c_1^2c_2t - 60z'c_1c_2^2t^2 + 12z'c_0c_1c_2 + 24z'c_0c_2^2t - 18c_0c_1^2t^2c_2 - 24c_0c_1t^3c_2^2 + 4z''^2c_1c_2t + 6z''c_1^2t^2c_2 + 8z''c_1t^3c_2^2 - 4z'z''c_1c_2 - 8z'z''c_2^2t - 24c_1t^5c_2^3 + 2z''c_0c_1^2 - 4z''c_0^2c_2 + 4z''^2c_2^2t^2 + 2z''c_1^3t + 4z''c_2^3t^4 \neq 0$ and $G(u^*; P(u^*, \bar{y})) = 0$. Regard $G(u^*; z)$ as an algebraic polynomial in $\mathbb{Q}(t)[c_0, c_1, c_2; z, z', z'']$, denoted by $g(c_0, c_1, c_2; z, z', z'')$. Then $P(u^*, \bar{y}), P(u^*, \bar{y})', P(u^*, \bar{y})''$ are algebraically dependent over $\mathbb{Q}(t)(c_0, c_1, c_2)$. So the problem is converted to an algebraic one, and we can use Lemma 2.13 to solve it.

To be more precise, first, specialize c_2 to 0; we obtain $g(c_0, c_1, 0; z, z', z'') = z''^2c_1^2 + 2z''c_0c_1^2 + 2z''c_1^3t - 6c_0c_1^3t - 8z'c_1^3 - 3c_0^2c_1^2 - 3c_1^4t^2 \neq 0$ and $g(c_0, c_1, 0; P(c_0 + c_1t, \bar{y}), P(c_0 + c_1t, \bar{y})', P(c_0 + c_1t, \bar{y})'') = 0$. Then specialize c_1 to 0; we obtain $g(c_0, 0, 0; z, z', z'') = 0$ and $g(c_0, c_1, 0; z, z', z'') = c_1^2g_1$, where $g_1 = z''^2 + 2z''c_0 + 2z''c_1t - 6c_0c_1t - 8z'c_1 - 3c_0^2 - 3c_1^2t^2$. Clearly, $g_1(c_0, c_1; P(c_0 + c_1t, \bar{y}), P(c_0 + c_1t, \bar{y})', P(c_0 + c_1t, \bar{y})'') = 0$. Specialize c_1 to 0 in g_1 , $\bar{g}_1 = z''^2 + 2z''c_0 - 3c_0^2$, while $\bar{g}_1(c_0; P(c_0, \bar{y}), P(c_0, \bar{y})', P(c_0, \bar{y})'') = 0$. Now specialize c_0 to $\bar{u} = 1$ in \bar{g}_1 ; we obtain $g_2(z, z', z'') = z''^2 + 2z'' - 3$. Clearly, $g_2(\xi, \xi', \xi'') = 0$, where $\xi = P(\bar{u}, \bar{y})$. Thus, $P(\bar{u}, \bar{y})$ is δ -algebraic over $\mathbb{Q}(t)$. Also, $P(\bar{u}, \bar{y}), P(\bar{u}, \bar{y})'$ and $P(\bar{u}, \bar{y})''$ are algebraically dependent over $\mathbb{Q}(t)$.

3. INTERSECTION THEORY FOR GENERIC DIFFERENTIAL POLYNOMIALS

In this section, we will develop an intersection theory for generic δ -polynomials by proving Theorem 1.1. As a consequence, the dimension conjecture is shown to be true for generic δ -polynomials. These results will also be used in Sections 4 and 6 to determine the order of the Chow form.

3.1. Generic dimension theorem. In this section, we will show that the dimension conjecture is valid for certain generic δ -polynomials. To prove the dimension conjecture in the general case, one simple idea is to generalize the following theorem ([19, p.43]) in algebra to the differential case.

Theorem 3.1. *Let \mathcal{I} be a prime ideal of dimension $d > 0$ and $f \in \mathcal{F}[\mathbb{Y}]$. If $(\mathcal{I}, f) \neq (1)$, then every prime component of $\mathbb{V}(\mathcal{I}, f)$ has dimension not less than $d - 1$. Moreover, if f is not in \mathcal{I} , then every prime component of $\mathbb{V}(\mathcal{I}, f)$ has dimension $d - 1$.*

Unfortunately, in the differential case, the above theorem does not hold. Ritt gave the following counterexample.

Example 3.2 ([32, p.133]). $p = y_1^5 - y_2^5 + y_3(y_1y_2' - y_2y_1')^2 \in \mathcal{F}\{y_1, y_2, y_3\}$ and $f = y_3$, where \mathcal{F} is the field of complex numbers. Then $\text{sat}(p)$ is a prime δ -ideal of dimension two. But $\sqrt{[\text{sat}(p), f]} = [y_1, y_2, y_3]$, which is a prime δ -ideal of dimension zero.

It could also happen that when adding a δ -polynomial to a prime δ -ideal to which it does not belong, the dimension is still the same.

Example 3.3. Let $p = y_1' y_2'' - y_1'' y_2'$. Then $\text{sat}(p) = [p] : y_1'^{\infty}$ is a prime δ -ideal of dimension one. It is clear that $y_2' \notin \text{sat}(p)$ and $[\text{sat}(p), y_2'] = [y_2']$ is still a prime δ -ideal of dimension one.

In this section, we will prove that Theorem 3.1 is valid for certain generic δ -polynomials, which will lead to the solution of the dimension conjecture in these generic cases.

Definition 3.4. Let $\mathfrak{m}_{s,r}$ be the set of all δ -monomials in $\mathcal{F}\{\mathbb{Y}\}$ of order $\leq s$ and degree $\leq r$. Let $\mathbb{U} = \{u_m\}_{m \in \mathfrak{m}_{s,r}}$ be a set of elements of \mathcal{E} that are δ - \mathcal{F} -indeterminates. Then,

$$f = \sum_{m \in \mathfrak{m}_{s,r}} u_m m$$

is called a *generic δ -polynomial* of order s and degree r . If $s = 0$, then f is also called a *generic polynomial* of degree r . A *generic δ -hypersurface* is the set of zeros of a generic δ -polynomial.

We use \mathbf{u}_f to denote the set of coefficients of a generic δ -polynomial f and

$$(3.1) \quad \tilde{\mathbf{u}}_f = \mathbf{u}_f \setminus \{u_0\},$$

where u_0 is the degree zero term of f . By saying that a point $\eta \in \mathcal{E}^n$ is free from the pure δ -extension field $\mathcal{F}\langle \mathbf{u}_f \rangle$ over \mathcal{F} , we mean \mathbf{u}_f are δ - $\mathcal{F}\langle \eta \rangle$ -indeterminates.

Throughout the paper, a generic δ -polynomial is assumed to be of degree greater than zero.

Lemma 3.5. *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension d and f a generic δ -polynomial. Then $\mathcal{I}_0 = [\mathcal{I}, f]$ is a prime δ -ideal in $\mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle\{\mathbb{Y}, u_0\}$ with dimension d , where $\tilde{\mathbf{u}}_f$ is defined in (3.1). Furthermore, $\mathcal{I}_0 \cap \mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle\{u_0\} = \{0\}$ if and only if $d > 0$.*

Proof. Let $\xi = (\xi_1, \dots, \xi_n)$ be a generic point of \mathcal{I} over \mathcal{F} that is free from $\mathcal{F}\langle \mathbf{u}_f \rangle$ and $f = u_0 + f_0$, where $f_0 = \sum_{\deg(m) \geq 1} u_m m$. We claim that $(\xi_1, \dots, \xi_n, -f_0(\xi))$ is a generic point of \mathcal{I}_0 over $\mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle$. Thus, it follows that \mathcal{I}_0 is a prime δ -ideal by Lemma 2.1.

Clearly, $(\xi_1, \dots, \xi_n, -f_0(\xi))$ is a zero of \mathcal{I}_0 . Let g be any δ -polynomial in $\mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle\{y_1, \dots, y_n, u_0\}$ which vanishes at $(\xi_1, \dots, \xi_n, -f_0(\xi))$. Regarding u_0 as the leader of f , suppose the pseudo remainder of g w.r.t. f is g_1 . Then we have

$$g \equiv g_1, \text{ mod } [f],$$

where $g_1 \in \mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle\{y_1, \dots, y_n\}$. From the above expression, $g_1(\xi) = 0$. Since ξ is also a generic point of $\mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle \cdot \mathcal{I}$ over $\mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle$, $g_1 \in \mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle \cdot \mathcal{I}$. Thus, $g \in \mathcal{I}_0$, and it follows that $(\xi_1, \dots, \xi_n, -f_0(\xi))$ is a generic point of \mathcal{I}_0 over $\mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle$ and \mathcal{I}_0 is prime.

By Lemma 2.12,

$$\begin{aligned} \dim \mathcal{I}_0 &= \text{d.tr.deg } \mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle\langle \xi_1, \dots, \xi_n, -f_0(\xi) \rangle / \mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle \\ &= \text{d.tr.deg } \mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle\langle \xi_1, \dots, \xi_n \rangle / \mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle \\ &= \text{d.tr.deg } \mathcal{F}\langle \xi_1, \dots, \xi_n \rangle / \mathcal{F} = d. \end{aligned}$$

Now consider the second part of the lemma. If $d = 0$, then $\dim \mathcal{I}_0 = 0$, so $\mathcal{I}_0 \cap \mathcal{F}(\tilde{\mathbf{u}}_f)\{u_0\} \neq \{0\}$. Thus, if $\mathcal{I}_0 \cap \mathcal{F}(\tilde{\mathbf{u}}_f)\{u_0\} = \{0\}$, then $d > 0$. It remains to show that if $d > 0$, then $\mathcal{I}_0 \cap \mathcal{F}(\tilde{\mathbf{u}}_f)\{u_0\} = \{0\}$. Suppose the contrary; then there exists a nonzero δ -polynomial $p(\tilde{\mathbf{u}}_f, u_0) \in \mathcal{I}_0 \cap \mathcal{F}(\tilde{\mathbf{u}}_f, u_0)$. So $p(\tilde{\mathbf{u}}_f, -f_0(\xi)) = 0$. Then, $\phi = -f_0(\xi)$ is δ -algebraic over $\mathcal{F}(\tilde{\mathbf{u}}_f)$. Denote the coefficient of any monomial y_i in f to be u_{i0} . So for any fixed i when u_{i0} is specialized to -1 and all the other $u \in \tilde{\mathbf{u}}_f$ specialized to zero, by Theorem 2.16, we conclude that $\bar{\phi} = \xi_i (i = 1, \dots, n)$ is δ -algebraic over \mathcal{F} , which contradicts the fact that \mathcal{I} has a positive dimension. So $\mathcal{I}_0 \cap \mathcal{F}(\tilde{\mathbf{u}}_f)\{u_0\} = \{0\}$. □

Now, we will prove the first key result of this paper, which shows that by adding a generic δ -polynomial to a prime δ -ideal, the new δ -ideal is still prime and its dimension decreases by one. This is generally not valid if the δ -polynomial is not generic, as shown in Examples 3.2 and 3.3.

Theorem 3.6. *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension d and f a generic δ -polynomial. If $d > 0$, then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime δ -ideal in $\mathcal{F}(\mathbf{u}_f)\{\mathbb{Y}\}$ with dimension $d - 1$. Also, if $d = 0$, then \mathcal{I}_1 is the unit ideal in $\mathcal{F}(\mathbf{u}_f)\{\mathbb{Y}\}$.*

Proof. First, we consider the case $d > 0$. Let (ξ_1, \dots, ξ_n) be a generic point of \mathcal{I} over \mathcal{F} that is free from the pure δ -extension field $\mathcal{F}(\mathbf{u}_f)$ over \mathcal{F} and $f = u_0 + f_0$ where $f_0 = \sum_{\deg(m) \geq 1} u_m m$. By Lemma 3.5, $\mathcal{I}_0 = [\mathcal{I}, f] \subset \mathcal{F}(\tilde{\mathbf{u}}_f)\{\mathbb{Y}, u_0\}$ is a prime δ -ideal, where $\tilde{\mathbf{u}}_f$ is defined in (3.1). By Lemma 3.5, $\mathcal{I}_0 \cap \mathcal{F}(\tilde{\mathbf{u}}_f)\{u_0\} = \{0\}$. So $\mathcal{I}_1 = [\mathcal{I}_0]$ in $\mathcal{F}(\mathbf{u}_f)\{\mathbb{Y}\}$ is not the unit ideal, for if not, $1 \in \mathcal{I}_1$. Then there exist δ -polynomials $g_i \in \mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$, $H_{ij}, G_k \in \mathcal{F}(\mathbf{u}_f)\{\mathbb{Y}\}$ such that $1 = \sum_{i,j} H_{ij} g_i^{(j)} + \sum_k G_k f^{(k)}$. By clearing denominators in the above expression, we obtain a nonzero δ -polynomial in $\mathcal{I}_0 \cap \mathcal{F}(\mathbf{u}_f)$, a contradiction.

Now we claim that \mathcal{I}_1 is prime and $\mathcal{I}_1 \cap \mathcal{F}(\tilde{\mathbf{u}}_f)\{\mathbb{Y}, u_0\} = \mathcal{I}_0$. Suppose $g, h \in \mathcal{F}(\mathbf{u}_f)\{\mathbb{Y}\}$ and $gh \in \mathcal{I}_1$. By collecting denominators, there exist $D_1, D_2 \in \mathcal{F}(\mathbf{u}_f)$ such that $D_1 g, D_2 h \in \mathcal{F}(\mathbf{u}_f, \mathbb{Y})$, and $(D_1 g) \cdot (D_2 h) = D_1 D_2 (gh) \in \mathcal{I}_1$ still holds. Similar to the procedure above, we can find a nonzero $D \in \mathcal{F}(\mathbf{u}_f)$ such that $D \cdot (D_1 g) \cdot (D_2 h) \in \mathcal{I}_0$. Since \mathcal{I}_0 is prime and $\mathcal{I}_0 \cap \mathcal{F}(\tilde{\mathbf{u}}_f)\{u_0\} = \{0\}$, $D_1 g \in \mathcal{I}_0$ or $D_2 h \in \mathcal{I}_0$. It follows that $g \in \mathcal{I}_1$ or $h \in \mathcal{I}_1$. Since \mathcal{I}_1 is not the unit ideal, \mathcal{I}_1 is prime. Also, for any $g \in \mathcal{I}_1 \cap \mathcal{F}(\tilde{\mathbf{u}}_f)\{\mathbb{Y}, u_0\}$, there exists some $D \in \mathcal{F}(\mathbf{u}_f)$ such that $Dg \in \mathcal{I}_0$, so $g \in \mathcal{I}_0$. Thus, $\mathcal{I}_1 \cap \mathcal{F}(\tilde{\mathbf{u}}_f)\{\mathbb{Y}, u_0\} = \mathcal{I}_0$.

Suppose ξ_1, \dots, ξ_d are δ -independent over \mathcal{F} . Then, $\{y_1, \dots, y_d\}$ is a parametric set of \mathcal{I} . Thus each $y_{d+i} (i = 1, \dots, n - d)$ is δ -dependent with y_1, \dots, y_d modulo \mathcal{I}_1 , since $\mathcal{I} \subset \mathcal{I}_1$. By Lemma 3.5, $\dim \mathcal{I}_0 = d$. Then u_0, y_1, \dots, y_d are δ -dependent modulo \mathcal{I}_0 , so $\{y_1, \dots, y_d\}$ is δ -dependent modulo \mathcal{I}_1 . Thus $\dim \mathcal{I}_1 \leq d - 1$. Now we claim y_1, \dots, y_{d-1} are δ -independent modulo \mathcal{I}_1 , which proves $\dim \mathcal{I}_1 = d - 1$. Suppose to the contrary that y_1, \dots, y_{d-1} are δ -dependent modulo \mathcal{I}_1 . Thus there exists a nonzero δ -polynomial $p(y_1, \dots, y_{d-1}) \in \mathcal{I}_1$. Take $p \in \mathcal{F}(\tilde{\mathbf{u}}_f, y_1, \dots, y_{d-1}, u_0)$. Then

$$p(\tilde{\mathbf{u}}_f, \xi_1, \dots, \xi_{d-1}, -f_0(\xi)) = 0.$$

That is, $\xi_1, \dots, \xi_{d-1}, -f_0(\xi)$ are δ -dependent over $\mathcal{F}(\tilde{\mathbf{u}}_f)$. Now we specialize u_{d0} to -1 , and the other $u \in \tilde{\mathbf{u}}_f$ to zero, where u_{d0} refers to the coefficient of the monomial y_d in f . Then $-f_0(\xi)$ is specialized to ξ_d . By Theorem 2.16, ξ_1, \dots, ξ_d are δ -dependent over \mathcal{F} , which is a contradiction. So in this case $\dim \mathcal{I}_1 = d - 1$.

Now, it remains to show the case $d = 0$. Since $d = 0$, by Lemma 3.5, $\mathcal{I}_0 \cap \mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle \{u_0\} \neq \{0\}$. So $\mathcal{I}_0 \cap \mathcal{F}\langle \mathbf{u}_f \rangle \neq \{0\}$, and consequently $\mathcal{I}_1 = [\mathcal{I}_0]$ in $\mathcal{F}\langle \mathbf{u}_f \rangle \{\mathbb{Y}\}$ is the unit ideal. \square

A special case of Theorem 3.6 is particularly interesting, and its algebraic counterpart is often listed as a theorem in algebraic geometry textbooks [16, p.54, p.110].

Theorem 3.7. *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension $d > 0$. Let $\{u_0, u_1, \dots, u_n\} \subset \mathcal{E}$ be a set of δ - \mathcal{F} -indeterminates. Then $\mathcal{I}_1 = [\mathcal{I}, u_0 + u_1y_1 + \dots + u_ny_n]$ is a prime δ -ideal in $\mathcal{F}\langle u_0, u_1, \dots, u_n \rangle \{\mathbb{Y}\}$ with dimension $d - 1$.*

Theorem 3.6 is also valid for a wider class of δ -polynomials. A δ -polynomial f in \mathbb{Y} is said to be *quasi-generic* if 1) the coefficients of f as a δ -polynomial in y_1, \dots, y_n are δ -indeterminates and 2) in addition to the degree zero term, for each $1 \leq i \leq n$, f also contains at least one δ -monomial in $\mathcal{F}\{y_i\} \setminus \mathcal{F}$. For instance, $f = u_0 + u_1y_1 + u_2y_1y_2$ is not quasi-generic, because f contains no monomials in $\mathcal{F}\{y_2\} \setminus \mathcal{F}$.

The proof for Theorem 3.6 can be easily adapted to prove the following result.

Corollary 3.8. *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension d and f a quasi-generic δ -polynomial with \mathbf{u}_f as the set of coefficients. If $d > 0$, then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime δ -ideal in $\mathcal{F}\langle \mathbf{u}_f \rangle \{\mathbb{Y}\}$ with dimension $d - 1$. Also, if $d = 0$, then \mathcal{I}_1 is the unit ideal in $\mathcal{F}\langle \mathbf{u}_f \rangle \{\mathbb{Y}\}$.*

By *independent generic (resp. quasi-generic) δ -polynomials*, we mean that the coefficients of all of them are δ -independent over \mathcal{F} . As a direct consequence, we can show that the dimension conjecture is valid for quasi-generic δ -polynomials.

Theorem 3.9 (Generic Dimension Theorem). *Let f_1, \dots, f_r be independent quasi-generic δ -polynomials in $\mathcal{F}\langle \mathbf{u} \rangle \{\mathbb{Y}\}$ with $r \leq n$ and \mathbf{u} the set of coefficients of all f_i . Then $[f_1, \dots, f_r] \subset \mathcal{F}\langle \mathbf{u} \rangle \{\mathbb{Y}\}$ is a prime δ -ideal with dimension $n - r$. Also, if $r > n$, $[f_1, \dots, f_r]$ is the unit ideal.*

Proof. We prove the theorem by induction. Let $\mathcal{I} = [0]$. When $r = 1$, by Corollary 3.8, $[f_1]$ is prime with dimension $n - 1$. Assuming this holds for $r - 1$, now consider the case $r \leq n$. By the hypothesis, $[f_1, \dots, f_{r-1}]$ is a prime δ -ideal with dimension $n - r + 1$. Note that the coefficients of f_r are δ -indeterminates over $\mathcal{F}\langle \mathbf{u}_{f_1}, \dots, \mathbf{u}_{f_{r-1}} \rangle$. Using Corollary 3.8 again, $[f_1, \dots, f_r]$ is a prime δ -ideal with dimension $n - r$. When $r > n$, since $[f_1, \dots, f_n]$ is of dimension zero, by Corollary 3.8, $[f_1, \dots, f_r]$ is the unit ideal. \square

3.2. Order of a system of generic differential polynomials. In this section, we consider the order of the intersection of a δ -variety by a generic δ -hypersurface. Before proving the main result, we give a series of lemmas and theorems.

Lemma 3.10. *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension $n - 1$. Suppose $\{f\}$ is a characteristic set of \mathcal{I} w.r.t. some ranking \mathcal{R} and f is irreducible. Then for any other ranking $\overline{\mathcal{R}}$, $\{f\}$ is also a characteristic set of \mathcal{I} .*

Proof. Denote f to be \overline{f} under the ranking $\overline{\mathcal{R}}$. By Theorem 2.2, $\mathcal{I} = \text{sat}(f)$ and $\overline{\mathcal{I}} = \text{sat}(\overline{f})$ are prime δ -ideals with f and \overline{f} as characteristic sets, respectively. We need to show that $\mathcal{I} = \overline{\mathcal{I}}$. Let S be the separant of f . Then for $g \in \text{sat}(f)$, we have

$S^m g = hf + h_1 f' + \dots + h_s f^{(s)}$ for $m, s \in \mathbb{N}$. Then, $S^m g \in \text{sat}(\bar{f})$. Since $\text{sat}(\bar{f})$ is prime, we need only show that S is not in $\text{sat}(\bar{f})$. Suppose the contrary, $S \in \text{sat}(\bar{f})$. Since S is partially reduced w.r.t. \bar{f} , we have $S = h\bar{f}$ for a δ -polynomial h , which is impossible since $S = \frac{\partial f}{\partial u_f}$. So $\mathcal{I} \subseteq \bar{\mathcal{I}}$. Similarly, we can prove that $\mathcal{I} \supseteq \bar{\mathcal{I}}$, thus $\mathcal{I} = \bar{\mathcal{I}}$. □

If \mathcal{S} is any set of δ -polynomials in $\mathcal{E}\{\mathbb{Y}\}$, then its set of zeros in \mathcal{E}^n is called the δ -variety of \mathcal{S} , still denoted by $\mathbb{V}(\mathcal{S})$. The following lemma generalizes a result in [8, p.5] to the case of positive dimensions.

Lemma 3.11. *Let the δ -variety of a system \mathcal{S} of δ -polynomials in $\mathcal{F}\{\mathbb{Y}\}$ have a component V of dimension d and order h . Let $\bar{\mathcal{S}}$ be obtained from \mathcal{S} by replacing $y_1^{(k)}$ by $y_1^{(k+1)}$ ($k = 0, 1, \dots$) in all of the δ -polynomials of \mathcal{S} . Then the δ -variety of $\bar{\mathcal{S}}$ has a component \bar{V} of dimension d and order h_1 such that $h \leq h_1 \leq h + 1$. Moreover, if there exists a parametric set \mathbb{U} not containing y_1 such that the relative order of $\mathbb{I}(V)$ w.r.t. \mathbb{U} is h , then the order of \bar{V} is $h + 1$; otherwise, the order of \bar{V} is h . In particular, if $d = 0$, then $\text{ord}(\bar{V}) = h + 1$.*

Proof. Let (ξ_1, \dots, ξ_n) be a generic point of V and $\mathcal{I} = \mathbb{I}(V) \in \mathcal{F}\{\mathbb{Y}\}$. It is clear that $[z' - \xi_1]$ is a prime δ -ideal in $\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle\{z\}$. Let η be a generic point of $[z' - \xi_1]$. Then $(\eta, \xi_2, \dots, \xi_n)$ is a point of $\bar{\mathcal{S}}$. Suppose this point lies in a component \bar{V} of $\bar{\mathcal{S}}$, which has a generic point (η_1, \dots, η_n) . Then $(\eta_1, \eta_2, \dots, \eta_n)$ is specialized to $(\eta, \xi_2, \dots, \xi_n)$, and correspondingly $(\eta'_1, \eta_2, \dots, \eta_n)$ is specialized to $(\xi_1, \xi_2, \dots, \xi_n)$. Since (ξ_1, \dots, ξ_n) is a generic point of V and $(\eta'_1, \eta_2, \dots, \eta_n)$ is a zero of \mathcal{S} , the latter specialization is generic, that is, $(\eta'_1, \eta_2, \dots, \eta_n)$ is a generic point of V . We claim that any parametric set \mathbb{U} of \mathcal{I} is a parametric set of $\mathbb{I}(\bar{V})$, and $\text{ord}_{\mathbb{U}}\mathcal{I} \leq \text{ord}_{\mathbb{U}}\mathbb{I}(\bar{V}) \leq \text{ord}_{\mathbb{U}}\mathcal{I} + 1$, which follows that $\dim(\bar{V}) = d$, and by Theorem 2.11, $h \leq \text{ord}(\bar{V}) \leq h + 1$. Let \mathbb{U} be any parametric set of \mathcal{I} . We consider the following two cases.

Case 1: $y_1 \notin \mathbb{U}$. Suppose \mathbb{U} is the set of y_2, \dots, y_{d+1} . By Corollary 2.8 we have

$$\text{ord}_{y_2, \dots, y_{d+1}}\mathcal{I} = \text{tr.deg}\mathcal{F}\langle \xi_1, \dots, \xi_d, \xi_{d+1}, \dots, \xi_n \rangle / \mathcal{F}\langle \xi_2, \dots, \xi_{d+1} \rangle.$$

Since ξ_2, \dots, ξ_{d+1} are δ -independent over \mathcal{F} , $\eta_2, \dots, \eta_{d+1}$ must be δ -independent over \mathcal{F} , i.e. $\mathbb{I}(\bar{V}) \cap \mathcal{F}\{\mathbb{U}\} = \{0\}$.

$$\begin{aligned} & \text{tr.deg}\mathcal{F}\langle \eta_1, \eta_2, \dots, \eta_n \rangle / \mathcal{F}\langle \eta_2, \dots, \eta_{d+1} \rangle \\ \geq & \text{tr.deg}\mathcal{F}\langle \eta, \xi_2, \dots, \xi_n \rangle / \mathcal{F}\langle \xi_2, \dots, \xi_{d+1} \rangle \\ & \text{(for } (\eta_1, \eta_2, \dots, \eta_n) \text{ can be specialized to } (\eta, \xi_2, \dots, \xi_n)) \\ = & \text{tr.deg}\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle / \mathcal{F}\langle \xi_2, \dots, \xi_{d+1} \rangle + \text{tr.deg}\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle / \mathcal{F}\langle \xi_1, \dots, \xi_n \rangle \\ = & \text{ord}_{y_2, \dots, y_{d+1}}\mathcal{I} + 1 \end{aligned}$$

and

$$\begin{aligned} & \text{tr.deg}\mathcal{F}\langle \eta_1, \dots, \eta_n \rangle / \mathcal{F}\langle \eta_2, \dots, \eta_{d+1} \rangle \\ \leq & 1 + \text{tr.deg}\mathcal{F}\langle \eta'_1, \eta_2, \dots, \eta_n \rangle / \mathcal{F}\langle \eta_2, \dots, \eta_{d+1} \rangle \\ = & 1 + \text{ord}_{y_2, \dots, y_{d+1}}\mathcal{I}, \end{aligned}$$

so $\text{tr.deg}\mathcal{F}\langle \eta_1, \dots, \eta_n \rangle / \mathcal{F}\langle \eta_2, \dots, \eta_{d+1} \rangle = 1 + \text{ord}_{y_2, \dots, y_{d+1}}\mathcal{I} < \infty$. Thus \bar{V} is of dimension d and $\{y_2, \dots, y_{d+1}\}$ is a parametric set of $\mathbb{I}(\bar{V})$. Moreover, the relative order of $\mathbb{I}(\bar{V})$ w.r.t. y_2, \dots, y_{d+1} is $\text{ord}_{y_2, \dots, y_{d+1}}\mathcal{I} + 1$.

Case 2: $y_1 \in \mathbb{U}$. Suppose $\mathbb{U} = \{y_1, \dots, y_d\}$. Then by Corollary 2.8, $\text{ord}_{\mathbb{U}}\mathcal{I} = \text{ord}_{y_1, \dots, y_d}\mathcal{I} = \text{tr.deg } \mathcal{F}\langle \xi_1, \dots, \xi_n \rangle / \mathcal{F}\langle \xi_1, \dots, \xi_d \rangle$. We have seen that $(\eta'_1, \eta_2, \dots, \eta_n)$ is a generic point of V . Since $\text{tr.deg } \mathcal{F}\langle \xi_1, \dots, \xi_n \rangle \langle \eta \rangle / \mathcal{F}\langle \xi_1, \dots, \xi_n \rangle = 1$ and $(\eta_1, \eta_2, \dots, \eta_n)$ can be specialized to $(\eta, \xi_2, \dots, \xi_n)$, η_1 is algebraically independent over $\mathcal{F}\langle \eta'_1, \eta_2, \dots, \eta_n \rangle$. So

$$\begin{aligned} & \text{tr.deg } \mathcal{F}\langle \eta_1, \dots, \eta_n \rangle / \mathcal{F}\langle \eta_1, \dots, \eta_d \rangle \\ &= \text{tr.deg } \mathcal{F}(\eta_1) \langle \eta'_1, \eta_2, \dots, \eta_n \rangle / \mathcal{F}(\eta_1) \langle \eta'_1, \eta_2, \dots, \eta_d \rangle \\ &= \text{tr.deg } \mathcal{F}\langle \eta'_1, \eta_2, \dots, \eta_n \rangle / \mathcal{F}\langle \eta'_1, \eta_2, \dots, \eta_d \rangle \\ & \quad (\text{for } \text{tr.deg } \mathcal{F}\langle \eta'_1, \dots, \eta_n \rangle (\eta_1) / \mathcal{F}\langle \eta'_1, \dots, \eta_n \rangle = 1) \\ &= \text{ord}_{y_1, \dots, y_d}\mathcal{I}. \end{aligned}$$

Since (η_1, \dots, η_d) can be specialized to $(\eta, \xi_2, \dots, \xi_d)$ over \mathcal{F} , $d \geq \text{d.tr.deg } \mathcal{F}\langle \eta_1, \dots, \eta_d \rangle / \mathcal{F} \geq \text{d.tr.deg } \mathcal{F}\langle \eta, \xi_2, \dots, \xi_d \rangle / \mathcal{F} \geq \text{d.tr.deg } \mathcal{F}\langle \xi_1, \xi_2, \dots, \xi_d \rangle / \mathcal{F} = d$. Since $\text{tr.deg } \mathcal{F}\langle \eta_1, \dots, \eta_n \rangle / \mathcal{F}\langle \eta_1, \dots, \eta_d \rangle < \infty$, we have $\text{d.tr.deg } \mathcal{F}\langle \eta_1, \dots, \eta_d, \dots, \eta_n \rangle / \mathcal{F} = d$. Thus in this case, $\dim(\overline{V}) = d$, and $\mathbb{U} = \{y_1, \dots, y_d\}$ is a parametric set of $\mathbb{I}(\overline{V})$ with $\text{ord}_{y_1, \dots, y_d}\mathbb{I}(\overline{V}) = \text{ord}_{y_1, \dots, y_d}\mathcal{I}$.

Considering the two cases together, we can see $\dim(\overline{V}) = d$. Also, by Theorem 2.11, $h \leq \text{ord}(\overline{V}) \leq h + 1$. Moreover, if there exists a parametric set \mathbb{U} not containing y_1 such that the relative order of $\mathbb{I}(V)$ w.r.t. \mathbb{U} is h , then the order of \overline{V} is $h + 1$; otherwise, the order of \overline{V} is h . In particular, if $d = 0$, then $y_1 \notin \mathbb{U} = \emptyset$. From Case 1, $\text{ord}(\overline{V}) = \text{ord}(V) + 1 = h + 1$. □

Let $\mathcal{G} \subset \mathcal{E}$ be a δ -extension field of \mathcal{F} . By a δ - \mathcal{F} -isomorphism of \mathcal{G} , we mean a δ -isomorphic mapping of \mathcal{G} onto a δ -field $\mathcal{G}' \subset \mathcal{E}$ such that (a) \mathcal{G}' is an extension of \mathcal{F} , (b) the δ -isomorphic mapping leaves each element of \mathcal{F} invariant. By means of well-ordering methods, it is easy to show that a δ - \mathcal{F} -isomorphism of \mathcal{G} can be extended to a δ - \mathcal{F} -automorphism of \mathcal{E} . We will use the following result regarding δ -isomorphisms.

Theorem 3.12 ([20]). *Let $\mathcal{G} \subset \mathcal{E}$ be a δ -extension field of \mathcal{F} and $\gamma \in \mathcal{G}$. A necessary and sufficient condition that γ be a primitive element of \mathcal{G} , i.e. $\mathcal{G} = \mathcal{F}\langle \gamma \rangle$, is that no \mathcal{F} -isomorphism of \mathcal{G} other than the identity leaves γ invariant.*

The following theorem as well as Theorem 3.6 prove Theorem 1.1.

Theorem 3.13. *Let \mathcal{I} be a prime δ - \mathcal{F} -ideal with dimension $d > 0$ and order h , and f a generic δ -polynomial of order s . Then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime δ -ideal in $\mathcal{F}\langle \mathbf{u}_f \rangle \{ \mathbb{Y} \}$ with dimension $d - 1$ and order $h + s$.*

Proof. By Theorem 3.6, \mathcal{I}_1 is prime with dimension $d - 1$. Now we prove the order of \mathcal{I}_1 is $h + s$.

Let \mathcal{A} be a characteristic set of \mathcal{I} w.r.t. an orderly ranking \mathcal{R} with y_1, \dots, y_d as a parametric set. By Theorem 2.6, $\text{ord}(\mathcal{A}) = h$. Suppose $\xi = (\xi_1, \dots, \xi_n)$ is a generic point of \mathcal{I} that is free from the pure extension field $\mathcal{F}\langle \mathbf{u}_f \rangle$ over \mathcal{F} . Let $f = u_0 + f_0$, where $f_0 = \sum_{\text{deg}(m) \geq 1} u_m m$. Let $\mathcal{I}_0 = [\mathcal{I}, f]$ in $\mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle \{ y_1, \dots, y_n, u_0 \}$, where $\tilde{\mathbf{u}}_f$ is defined in (3.1). By Lemma 3.5, \mathcal{I}_0 is a prime δ -ideal of dimension d

with a generic zero $(\xi_1, \dots, \xi_n, -f_0(\xi))$, and u_0 is δ -independent modulo \mathcal{I}_0 . \mathcal{I}_0 and \mathcal{I}_1 have such relations: Any characteristic set of \mathcal{I}_0 with u_0 in the parametric set is a characteristic set of \mathcal{I}_1 , and conversely, any characteristic set of \mathcal{I}_1 , by clearing denominators in $\mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle \{u_0\}$ when necessary, is a characteristic set of \mathcal{I}_0 with u_0 in the parametric set. By Theorem 2.11, we have $\text{ord}(\mathcal{I}_1) \leq \text{ord}(\mathcal{I}_0)$.

We claim that $\text{ord}(\mathcal{I}_0) \leq h + s$. As a consequence, $\text{ord}(\mathcal{I}_1) \leq h + s$. To prove this claim, let $\mathcal{I}_0^{(i)} = [\mathcal{I}, u_0^{(i)} + f_0]$ ($i = 0, \dots, s$) in $\mathcal{F}\langle \tilde{\mathbf{u}}_f \rangle \{y_1, \dots, y_n, u_0\}$. Similar to the proof of Lemma 3.5, $\mathcal{I}_0^{(i)}$ is a prime δ -ideal of dimension d . Let \bar{f} be the pseudo remainder of $u_0^{(s)} + f_0$ w.r.t. \mathcal{A} under the ranking \mathcal{R} . Clearly, $\text{ord}(\bar{f}, u_0) = s$. It is obvious that for some orderly ranking, $\{\mathcal{A}, \bar{f}\}$ is a characteristic set of $\mathcal{I}_0^{(s)}$ with y_1, \dots, y_d as a parametric set. So $\text{ord}(\mathcal{I}_0^{(s)}) = h + s$. Using Lemma 3.11 s times, we have $\text{ord}(\mathcal{I}_0) \leq \text{ord}(\mathcal{I}_0^{(1)}) \leq \dots \leq \text{ord}(\mathcal{I}_0^{(s)}) = h + s$.

Now, it suffices to show $\text{ord}(\mathcal{I}_1) \geq h + s$. Let $w = u_0 + \sum_{i=1}^d \sum_{j=0}^s u_{ij} y_i^{(j)}$ be a new δ -indeterminate. Let \mathbf{u}_g be the set of coefficients of $g = w + \sum_{i=d+1}^n \sum_{j=0}^s u_{ij} y_i^{(j)} + f_1$ regarded as a δ -polynomial in w and \mathbb{Y} , where f_1 is the nonlinear part of f in \mathbb{Y} . We denote $\mathcal{F}_1 = \mathcal{F}\langle \mathbf{u}_g \rangle$. Then $\mathcal{I}_2 = [\mathcal{I}, g]$ in $\mathcal{F}_1\{y_1, \dots, y_n, w\}$ is a prime δ -ideal with a generic point $(\xi_1, \dots, \xi_n, \gamma)$, where $\gamma = -\sum_{i=d+1}^n \sum_{j=0}^s u_{ij} \xi_i^{(j)} - f_1(\xi_1, \dots, \xi_n)$. We claim that γ is a primitive element of $\mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle \langle \xi_{d+1}, \dots, \xi_n \rangle$ over $\mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle$. By Theorem 3.12, it suffices to show that no $\mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle$ -isomorphism of $\mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle \langle \xi_{d+1}, \dots, \xi_n \rangle$ other than the identity leaves γ invariant. Let φ be any δ - $\mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle$ -isomorphism of $\mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle \langle \xi_{d+1}, \dots, \xi_n \rangle$ which leaves γ invariant, and $\varphi(\xi_{d+i}) = \eta_{d+i}$ ($i = 1, \dots, n - d$). Since each ξ_{d+i} ($i = 1, \dots, n - d$) is δ -algebraic over $\mathcal{F}\langle \xi_1, \dots, \xi_d \rangle$ and φ is an isomorphism leaving each element of $\mathcal{F}\langle \xi_1, \dots, \xi_d \rangle$ invariant, we can see that each η_{d+i} ($i = 1, \dots, n - d$) is also δ -algebraic over $\mathcal{F}\langle \xi_1, \dots, \xi_d \rangle$. So, η_{d+i} ($i = 1, \dots, n - d$) are also free from $\mathcal{F}\langle \mathbf{u}_f \rangle$. From $\varphi(\gamma) = \gamma$, we have $-\sum_{i=d+1}^n \sum_{j=0}^s u_{ij} \eta_i^{(j)} - f_1(\xi_1, \dots, \xi_d, \eta_{d+1}, \dots, \eta_n) = -\sum_{i=d+1}^n \sum_{j=0}^s u_{ij} \xi_i^{(j)} - f_1(\xi_1, \dots, \xi_d, \xi_{d+1}, \dots, \xi_n)$, which can be rewritten as

$$\sum_{i=d+1}^n \sum_{j=0}^s u_{ij} (\xi_i^{(j)} - \eta_i^{(j)}) + f_1(\xi_1, \dots, \xi_n) - f_1(\xi_1, \dots, \xi_d, \eta_{d+1}, \dots, \eta_n) = 0.$$

Since \mathbf{u}_f are δ -indeterminates over $\mathcal{F}\langle \xi_1, \dots, \xi_n, \eta_{d+1}, \dots, \eta_n \rangle$, we have $\xi_i - \eta_i = 0$ ($i = d + 1, \dots, n$). So φ must be the identity map and the claim follows.

Since $\mathcal{F}_1\langle \xi_1, \dots, \xi_d, \xi_{d+1}, \dots, \xi_n \rangle = \mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle \langle \gamma \rangle$, γ is δ -algebraic over $\mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle$ and each $\xi_{d+i} \in \mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle \langle \gamma \rangle$ ($i = 1, \dots, n - d$). Let $R(\xi_1, \dots, \xi_d, w)$ be an irreducible δ -polynomial in $\mathcal{F}_1\langle \xi_1, \dots, \xi_d \rangle \{w\}$ annulling γ of the lowest order. By clearing denominators when necessary, suppose $R(y_1, \dots, y_d, w)$ is an irreducible δ -polynomial in $\mathcal{F}_1\{y_1, \dots, y_d, w\}$. Clearly, $R(y_1, \dots, y_d, w) \in \mathcal{I}_2$. Also, there exist $A_i \in \mathcal{I}_2$ with the form $A_i = P_i(y_1, \dots, y_d, w) y_{d+i} + Q_i(y_1, \dots, y_d, w)$ ($i = 1, \dots, n - d$), which are reduced w.r.t. R . Since $\mathcal{I}_2 \cap \mathcal{F}_1\{y_1, \dots, y_d, w\}$ is a d -dimensional prime δ -ideal, by Lemma 3.10, $\{R\}$ is its characteristic set w.r.t. any ranking. So for the elimination ranking $y_1 \prec \dots \prec y_d \prec w \prec y_{d+1} \prec \dots \prec y_n$, a characteristic set of \mathcal{I}_2 is $\{R(y_1, \dots, y_d, w), A_1, \dots, A_{n-d}\}$. Since $\mathcal{F}_1\langle \xi_1, \dots, \xi_n, \gamma \rangle = \mathcal{F}_1\langle \xi_1, \dots, \xi_n \rangle$, by Corollary 2.8, $\text{ord}_{y_1, \dots, y_d} \mathcal{I}_2 = \text{ord}_{y_1, \dots, y_d} (\mathcal{I}) = \text{ord}(\mathcal{A}) = h$. Thus, $\text{ord}(R, w) = h$.

Let $\mathbb{U} = \{u_{ij} : i = 1, \dots, d; j = 0, \dots, s\}$. In $\mathcal{F}_1\langle\mathbb{U}\rangle\{w, y_1, \dots, y_n\}$, \mathcal{I}_2 is also prime with $R(y_1, \dots, y_d, w), A_1, \dots, A_{n-d}$ as a characteristic set w.r.t. the elimination ranking $y_1 \prec \dots \prec y_d \prec w \prec y_{d+1} \prec \dots \prec y_n$. Let

$$\begin{aligned} \phi : \mathcal{F}_1\langle\mathbb{U}\rangle\{y_1, \dots, y_n, w\} &\longrightarrow \mathcal{F}_1\langle\mathbb{U}\rangle\{y_1, \dots, y_n, u_0\}, \\ w &u_0 + \sum_{i=1}^d \sum_{j=0}^s u_{ij} y_i^{(j)}, \\ y_i &y_i \end{aligned}$$

be a δ -homomorphism over $\mathcal{F}_1\langle\mathbb{U}\rangle$. Clearly, this is a δ -isomorphism which maps \mathcal{I}_2 to \mathcal{I}_0 . It is obvious that \mathcal{I}_0 has $\phi(R), \phi(A_1), \dots, \phi(A_{n-d})$ as a characteristic set w.r.t. the elimination ranking $y_1 \prec \dots \prec y_d \prec u_0 \prec y_{d+1} \prec \dots \prec y_n$ with $\text{rk}(\phi(A_i)) = y_{d+i}(i = 1, \dots, n - d)$. We claim that $\text{ord}(\phi(R), y_1) \geq h + s$. If $\text{ord}(R, y_1) \geq h + s$, rewrite R in the form $R = \sum_{\psi_\nu(w) \neq 1} p_\nu(y_1, \dots, y_d) \psi_\nu(w) + p(y_1, \dots, y_d)$, where $\psi_\nu(w)$ are monomials in w and its derivatives. Then

$$\begin{aligned} \phi(R) &= \sum_{\psi_\nu \neq 1} p_\nu(y_1, \dots, y_d) \psi_\nu(u_0 + \sum_{i=1}^d \sum_{j=0}^s u_{ij} y_i^{(j)}) + p(y_1, \dots, y_d) \\ &= \sum_{\psi_\nu \neq 1} p_\nu(y_1, \dots, y_d) \psi_\nu(u_0) + p(y_1, \dots, y_d) \\ &\quad + \text{terms involving } u_{ij} (i = 1, \dots, d; j = 0, \dots, s) \text{ and their derivatives.} \end{aligned}$$

Clearly, in this case we have $\text{ord}(\phi(R), y_1) \geq \max\{\text{ord}(p_\nu, y_1), \text{ord}(p, y_1)\} = \text{ord}(R, y_1) \geq h + s$. If $\text{ord}(R, y_1) < h + s$, rewrite R as a polynomial in $w^{(h)}$, that is, $R = I_l(w^{(h)})^l + I_{l-1}(w^{(h)})^{l-1} + \dots + I_0$. Then $\phi(R) = \phi(I_l)[(u_0 + \sum_{i=1}^d \sum_{j=0}^s u_{ij} y_i^{(j)})^{(h)}]^l + \phi(I_{l-1})[(u_0 + \sum_{i=1}^d \sum_{j=0}^s u_{ij} y_i^{(j)})^{(h)}]^{l-1} + \dots + \phi(I_0)$. Since $\text{ord}(\phi(I_k), y_1) < h + s$ ($k = 0, \dots, l$), we have exactly $\text{ord}(\phi(R), y_1) = h + s$. Thus, consider the two cases together, $\text{ord}(\phi(R), y_1) \geq h + s$.

Since $\mathcal{I}_0 \cap \mathcal{F}_1\langle\mathbb{U}\rangle\{y_1, \dots, y_d, u_0\}$ is a d -dimensional prime δ -ideal, by Lemma 3.10, $\{\phi(R)\}$ is its characteristic set w.r.t. any ranking, in particular, for the elimination ranking $u_0 \prec y_2 \prec \dots \prec y_d \prec y_1$. So w.r.t. the elimination ranking $u_0 \prec y_2 \prec \dots \prec y_d \prec y_1 \prec y_{d+1} \prec \dots \prec y_n$, $\{\phi(R), \phi(A_1), \dots, \phi(A_{n-d})\}$ is a characteristic set of \mathcal{I}_0 , thus a characteristic set of \mathcal{I}_1 . By Theorem 2.11, $\text{ord}(\mathcal{I}_1) \geq \text{ord}_{y_2, \dots, y_d} \mathcal{I}_1 \geq h + s$.

Thus, the order of \mathcal{I}_1 is $h + s$. □

As a consequence, Theorem 3.7 can be strengthened as follows.

Theorem 3.14. *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\langle\mathbb{Y}\rangle$ with dimension $d > 0$ and order h . Let $\{u_0, u_1, \dots, u_n\} \subset \mathcal{E}$ be a set of δ - \mathcal{F} -indeterminates. Then $\mathcal{I}_1 = [\mathcal{I}, u_0 + u_1 y_1 + \dots + u_n y_n]$ is a prime δ -ideal in $\mathcal{F}\langle u_0, u_1, \dots, u_n \rangle\langle\mathbb{Y}\rangle$ with dimension $d - 1$ and order h .*

As another consequence, the dimension theorem for generic δ -polynomials can be strengthened as follows.

Theorem 3.15. *Let f_1, \dots, f_r ($r \leq n$) be independent generic δ -polynomials with each f_i of order s_i . Then $[f_1, \dots, f_r]$ is a prime δ -ideal with dimension $n - r$ and order $\sum_{i=1}^r s_i$ over $\mathcal{F}\langle \mathbf{u}_{f_1}, \dots, \mathbf{u}_{f_r} \rangle$.*

Proof. We will prove the theorem by induction on r . Let $\mathcal{I} = [0] \subset \mathcal{F}\langle\mathbb{Y}\rangle$. Clearly, \mathcal{I} is a prime δ -ideal of dimension n and order 0. For $r = 1$, by Theorem 3.13, $[f_1] = [\mathcal{I}, f_1]$ is a prime δ -ideal of dimension $n - 1$ and order s_1 . So the assertion

holds for $r = 1$. Now suppose the assertion holds for $r - 1$; we now prove it for r . By the hypothesis, $\mathcal{I}_{r-1} = [f_1, \dots, f_{r-1}]$ is a prime δ -ideal of dimension $n - r + 1$ and order $\sum_{i=1}^{r-1} s_i$ over $\mathcal{F}\langle \mathbf{u}_{f_1}, \dots, \mathbf{u}_{f_{r-1}} \rangle$. Since f_1, \dots, f_r are independent generic δ -polynomials, using Theorem 3.13 again, $\mathcal{I}_r = [f_1, \dots, f_r]$ is a prime δ -ideal of dimension $n - r$ and order $\sum_{i=1}^r s_i$ over $\mathcal{F}\langle \mathbf{u}_{f_1}, \dots, \mathbf{u}_{f_r} \rangle$. Thus, the theorem is proved. \square

Remark 3.16. When f is a quasi-generic δ -polynomial, Theorem 3.13 may not be true. A counterexample is as follows. Let $\mathcal{I} = [y_2, \dots, y_n] \in \mathcal{F}\{\mathbb{Y}\}$ and $f = u_0 + u_1 y_1 + u_2 y_2'' + \dots + u_n y_n''$. Clearly, f is a quasi-generic δ -polynomial and $[\mathcal{I}, f]$ is a prime δ -ideal of dimension 0. But $\text{ord}([\mathcal{I}, f]) = \text{ord}(\mathcal{I}) = 0 \neq \text{ord}(\mathcal{I}) + \text{ord}(f) = 2$.

4. CHOW FORM FOR AN IRREDUCIBLE DIFFERENTIAL VARIETY

In this section, we define the differential Chow form and establish its properties by proving Theorem 1.2.

4.1. Definition of the differential Chow form. Throughout this section, we assume that $V \subset \mathcal{E}^n$ is an irreducible δ - \mathcal{F} -variety with dimension d and $\mathcal{I} = \mathbb{I}(V) \subset \mathcal{F}\{\mathbb{Y}\}$. Let

$$(4.1) \quad \hat{\mathbf{u}} = \{u_{ij} \mid (i = 0, \dots, d; j = 0, \dots, n)\}$$

be $(d + 1)(n + 1)$ δ - \mathcal{F} -indeterminates in \mathcal{E} .

By saying that a point $\eta \in \mathcal{E}^n$ is free from the pure δ -extension field $\mathcal{F}\langle \hat{\mathbf{u}} \rangle$ over \mathcal{F} , we mean $\hat{\mathbf{u}}$ are δ -indeterminates over $\mathcal{F}\langle \eta \rangle$. Let $\xi = (\xi_1, \dots, \xi_n)$ be a generic point of V which is free from $\mathcal{F}\langle \hat{\mathbf{u}} \rangle$ and $\zeta_0, \zeta_1, \dots, \zeta_d$ elements of $\mathcal{F}\langle \mathbf{u}, \xi_1, \dots, \xi_n \rangle$:

$$(4.2) \quad \zeta_\sigma = - \sum_{\rho=1}^n u_{\sigma\rho} \xi_\rho \quad (\sigma = 0, \dots, d),$$

where

$$\mathbf{u} = \{u_{ij} \mid (i = 0, \dots, d; j = 1, \dots, n)\}.$$

We will show that the δ -transcendence degree of ζ_0, \dots, ζ_d over $\mathcal{F}\langle \mathbf{u} \rangle$ is d .

Lemma 4.1. $\text{d.tr.deg } \mathcal{F}\langle \mathbf{u} \rangle \langle \zeta_0, \dots, \zeta_d \rangle / \mathcal{F}\langle \mathbf{u} \rangle = d$. Furthermore, if $d > 0$, ζ_1, \dots, ζ_d are δ -independent over $\mathcal{F}\langle \mathbf{u} \rangle$.

Proof. By Lemma 2.12, $\text{d.tr.deg } \mathcal{F}\langle \mathbf{u} \rangle \langle \xi_1, \dots, \xi_n \rangle / \mathcal{F}\langle \mathbf{u} \rangle = d$. Since the $d + 1$ elements ζ_0, \dots, ζ_d belong to $\mathcal{F}\langle \mathbf{u}, \xi_1, \dots, \xi_n \rangle$, they are δ -dependent over $\mathcal{F}\langle \mathbf{u} \rangle$. Then, we have $\text{d.tr.deg } \mathcal{F}\langle \mathbf{u} \rangle \langle \zeta_0, \dots, \zeta_d \rangle / \mathcal{F}\langle \mathbf{u} \rangle \leq d$. Thus, if $d = 0$, we have

$$\text{d.tr.deg } \mathcal{F}\langle \mathbf{u} \rangle \langle \zeta_0, \dots, \zeta_d \rangle / \mathcal{F}\langle \mathbf{u} \rangle = 0.$$

Now, suppose $d > 0$. We claim that ζ_1, \dots, ζ_d are δ -independent over $\mathcal{F}\langle \mathbf{u} \rangle$, thus it follows that $\text{d.tr.deg } \mathcal{F}\langle \mathbf{u} \rangle \langle \zeta_0, \dots, \zeta_d \rangle / \mathcal{F}\langle \mathbf{u} \rangle = d$. Suppose the contrary. Since $\zeta_i \in \mathcal{F}\langle \mathbf{u}, \xi_1, \dots, \xi_n \rangle$, when we specialize u_{ij} to $-\delta_{k_{ij}}$ ($j = 1, \dots, n, k_i \in \{1, \dots, n\}$), ζ_i will be specialized to ξ_{k_i} . Then from Theorem 2.16, we conclude that $\xi_{k_1}, \dots, \xi_{k_d}$ are δ -dependent over \mathcal{F} . Since we can choose k_1, \dots, k_d so that $\xi_{k_1}, \dots, \xi_{k_d}$ are δ -independent over \mathcal{F} , it amounts to a contradiction. Thus the claim is proved. \square

Let \mathbb{I}_ζ be the prime δ -ideal in the δ -polynomial ring $\mathcal{R} = \mathcal{F}(\mathbf{u})\{z_0, \dots, z_d\}$ having $\zeta = (\zeta_0, \dots, \zeta_d)$ as a generic point. By Lemma 4.1, the dimension of \mathbb{I}_ζ is d . By Theorem 2.9, the characteristic set of \mathbb{I}_ζ w.r.t. any ranking consists of an irreducible δ -polynomial $f(z_0, \dots, z_d)$ in \mathcal{R} and

$$(4.3) \quad \mathbb{I}_\zeta = \text{sat}(f).$$

Since the coefficients of $f(z_0, \dots, z_d)$ are elements in $\mathcal{F}(\mathbf{u})$, without loss of generality, we assume $f(\mathbf{u}; z_0, \dots, z_d)$ is an irreducible δ -polynomial in $\mathcal{F}\{\mathbf{u}; z_0, \dots, z_d\}$. We shall subsequently replace $\{z_0, \dots, z_d\}$ by $\{u_{00}, \dots, u_{d0}\} \subset \widehat{\mathbf{u}}$, and obtain

$$(4.4) \quad F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \dots, u_{d0}) \in \mathcal{F}\{\widehat{\mathbf{u}}\},$$

where $\widehat{\mathbf{u}}$ is from (4.1) and $\mathbf{u}_i = (u_{i0}, \dots, u_{in})$ for $i = 0, \dots, d$.

Definition 4.2 (Differential Chow form). The δ -polynomial defined in (4.4) is called the *differential Chow form* of V or the prime δ -ideal $\mathcal{I} = \mathbb{I}(V)$.

A *generic δ -hyperplane* is the zero set of $u_0 + u_1y_1 + \dots + u_ny_n = 0$, where u_i are δ - \mathcal{F} -indeterminates. The following result shows that the differential Chow form can be obtained by intersecting \mathcal{I} with $d + 1$ generic δ -hyperplanes.

Lemma 4.3. *Using the notation introduced above, let $\mathcal{I} = \mathbb{I}(V)$ and*

$$(4.5) \quad \mathbb{P}_i = u_{i0} + u_{i1}y_1 + \dots + u_{in}y_n \quad (i = 0, \dots, d),$$

where u_{ij} are from (4.1). Then $[\mathcal{I}, \mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_d]$ is a prime δ -ideal in $\mathcal{F}\langle \mathbf{u} \rangle\{u_{00}, u_{10}, \dots, u_{d0}, y_1, \dots, y_n\}$ and $[\mathcal{I}, \mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_d] \cap \mathcal{F}\langle \mathbf{u} \rangle\{u_{00}, u_{10}, \dots, u_{d0}\} = \text{sat}(f)$.

Proof. Similar to the proof of Lemma 3.5, it is easy to show that $[\mathcal{I}, \mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_d]$ is a prime δ -ideal with a generic zero (ζ, ξ) . Denote $[\mathcal{I}, \mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_d]$ by $\mathbb{I}_{\zeta, \xi}$. Then, $\mathbb{I}_{\zeta, \xi} \cap \mathcal{F}\langle \mathbf{u} \rangle\{u_{00}, u_{10}, \dots, u_{d0}\}$ is a prime δ -ideal with a generic zero ζ , which implies $\mathbb{I}_{\zeta, \xi} \cap \mathcal{F}\langle \mathbf{u} \rangle\{u_{00}, u_{10}, \dots, u_{d0}\} = \mathbb{I}_\zeta = \text{sat}(f)$. □

In the following context, we will denote $[\mathcal{I}, \mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_d]$ by $\mathbb{I}_{\zeta, \xi}$.

Remark 4.4. From Lemma 4.3, we have two observations. First, the Chow form for a δ -variety is independent of the generic point used in (4.2). The δ -ideals \mathbb{I}_ζ and $\mathbb{I}_{\zeta, \xi}$ are also independent of the choice of ξ . Second, we can compute the Chow form of V with the δ -elimination algorithms [1, 7, 32, 39, 14] if we know a set of finitely many generating δ -polynomials for V . Furthermore, given a characteristic set \mathcal{A} of $\mathbb{I}(V)$, we can also compute its Chow form. Indeed, from Lemma 4.3, it suffices to compute a characteristic set of $\mathbb{I}_{\zeta, \xi}$ w.r.t. a ranking $\mathbb{U} \ll \mathbb{Y}$ (elimination ranking between elements of $\mathbb{U} = \{u_{00}, \dots, u_{d0}\}$ and \mathbb{Y}). It is clear that $\mathbb{I}_{\zeta, \xi}$ has a characteristic set $\{\mathcal{A}, \mathbb{P}_0, \dots, \mathbb{P}_d\}$ w.r.t. a ranking $\mathbb{Y} \ll \mathbb{U}$. Then using the algorithms given by Boulier et al. [1] and Golubitsky et al. [14] for transforming a characteristic decomposition of a radical δ -ideal from one ranking to another, we can obtain the Chow form.

Example 4.5. Consider the case $n = 1$. By Theorem 2.9, a prime δ -ideal in $\mathcal{F}\{y_1\}$ is of the form $\text{sat}(p)$ where $p \in \mathcal{F}\{y_1\}$ is irreducible. The zero set of $\text{sat}(p)$ is an irreducible δ - \mathcal{F} -variety in the *affine line*: $\mathbb{A}^1(\mathcal{E})$. Let $\mathbb{P}_0 = u_{00} + u_{01}y_1$ and $\mathbf{u}_0 = (u_{00}, u_{01})$. Then the Chow form of $\text{sat}(p)$ is $F(\mathbf{u}_0) = u_{00}^d p(-\frac{u_{01}}{u_{00}})$, where d is a natural number such that $F(\mathbf{u}_0)$ is an irreducible δ -polynomial in $\mathcal{F}\{\mathbf{u}_0\}$. For instance, let $p = (y_1')^2 - 4y_1$. Then the Chow form of $\text{sat}(p)$ is $F(\mathbf{u}_0) = u_{01}^2 (u_{00}')^2 - 2u_{01}u_{01}'u_{00}u_{00}' + (u_{01}')^2u_{00}^2 + 4u_{01}^3u_{00}$.

Example 4.6. If V is an irreducible δ -variety of dimension $n - 1$ and its corresponding prime δ -ideal is $\mathcal{I} = \text{sat}(p) \subset \mathcal{F}\{\mathbb{Y}\}$, then its Chow form is $F(\mathbf{u}_0, \dots, \mathbf{u}_{n-1}) = D^m p(\frac{D_1}{D}, \dots, \frac{D_n}{D})$, where

$$D = \begin{vmatrix} u_{01} & u_{02} & \cdots & u_{0n} \\ u_{11} & u_{12} & \cdots & u_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ u_{n-1,1} & u_{n-1,2} & \cdots & u_{n-1,n} \end{vmatrix}$$

and $D_i (i = 1, \dots, n)$ is the determinant of the matrix formed by replacing the i -th column of D by the column vector $(-u_{00}, -u_{10}, \dots, -u_{n-1,0})^T$, and m is the minimal integer such that $D^m p(\frac{D_1}{D}, \dots, \frac{D_n}{D}) \in \mathcal{F}\{\mathbf{u}; u_{00}, \dots, u_{d0}\}$.

Example 4.7. Let V be the irreducible δ -variety corresponding to $[y'_1 + 1, y'_2] \in \mathbb{Q}(t)\{y_1, y_2\}$. It is of dimension zero and the Chow form of V is $F(\mathbf{u}_0) = u_{02}u'_{01}u''_{00} + u''_{01}u_{01}u_{02} - 2u_{02}(u'_{01})^2 - u_{02}u''_{01}u'_{00} - u'_{02}u_{01}u''_{00} - u''_{02}u_{01}^2 - u'_{01}u'_{02}u_{00} + 2u'_{02}u_{01}u'_{01} + u''_{02}u_{01}u'_{00} + u'_{01}u'_{02}u_{00}$, where $\mathbf{u}_0 = (u_{00}, u_{01}, u_{02})$.

4.2. The order of the differential Chow form. In this section, we will show that the order of the differential Chow form is the same as that of the corresponding δ -variety. Before this, we give the following lemmas.

Lemma 4.8. Let $\zeta_0, \zeta_1, \dots, \zeta_d$ be defined in (4.2), and $f(\mathbf{u}; u_{00}, \dots, u_{d0})$ be the Chow form of V . Then for any $p(\mathbf{u}; u_{00}, \dots, u_{d0}) \in \mathcal{F}(\mathbf{u})\{u_{00}, \dots, u_{d0}\}$ with $\text{ord}(f) = \text{ord}(p)$ such that $p(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$, we have $p(\mathbf{u}; u_{00}, \dots, u_{d0}) = f(\mathbf{u}; u_{00}, \dots, u_{d0})h(\mathbf{u}; u_{00}, \dots, u_{d0})$, where $h(\mathbf{u}; u_{00}, \dots, u_{d0})$ is in $\mathcal{F}(\mathbf{u})\{u_{00}, \dots, u_{d0}\}$.

Proof. Since $\{f\}$ is the characteristic set of \mathbb{I}_ζ w.r.t. an orderly ranking, and $p \in \mathbb{I}_\zeta$ with $\text{ord}(f) = \text{ord}(p)$, then $\mathbb{I}_f^m p = fh$ for some $m \in \mathbb{N}$. Since f is irreducible, we can see that f divides p . □

The Chow form $f(\mathbf{u}; u_{00}, \dots, u_{d0})$ has certain symmetric properties as shown by the following results.

Lemma 4.9. Let $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ be the Chow form of an irreducible δ - \mathcal{F} -variety V and $F^*(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ be obtained from F by interchanging \mathbf{u}_ρ and \mathbf{u}_τ . Then F^* and F differ at most by a sign. Furthermore, $\text{ord}(F, u_{ij}) (i = 0, \dots, d; j = 0, 1, \dots, n)$ are the same for all u_{ij} occurring in F . In particular, $u_{i0} (i = 0, \dots, d)$ appear effectively in F . A necessary and sufficient condition for some $u_{ij} (j > 0)$ not occurring effectively in F is $y_j \in \mathbb{I}(V)$.

Proof. Consider the δ -automorphism ϕ of $\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle \langle \mathbf{u} \rangle$ over $\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle$:

$$\phi(u_{ij}) = u^*_{ij} = \begin{cases} u_{ij}, & i \neq \rho, \tau, \\ u_{\tau j}, & i = \rho, \\ u_{\rho j}, & i = \tau. \end{cases}$$

Of course,

$$\phi(\zeta_i) = \zeta^*_i = \begin{cases} \zeta_i, & i \neq \rho, \tau, \\ \zeta_\tau, & i = \rho, \\ \zeta_\rho, & i = \tau. \end{cases}$$

Then $\phi(f(\mathbf{u}; \zeta_0, \dots, \zeta_\rho, \dots, \zeta_\tau, \dots, \zeta_d)) = f(\mathbf{u}^*; \zeta_0, \dots, \zeta_\tau, \dots, \zeta_\rho, \dots, \zeta_d) = 0$. Instead of $f(\mathbf{u}; z_0, \dots, z_d)$, we obtain another δ -polynomial $p(\mathbf{u}; z_0, \dots, z_d) = f(\mathbf{u}^*; z_0, \dots, z_\tau, \dots, z_\rho, \dots, z_d) \in \mathbb{I}_\zeta$. Since the two δ -polynomials f and p have the same order and degree, and as algebraic polynomials they have the same content, by Lemma

4.8, $f(\mathbf{u}^*; z_0, \dots, z_\tau, \dots, z_\rho, \dots, z_d)$ can only differ by a sign with $f(\mathbf{u}; z_0, \dots, z_\rho, \dots, z_\tau, \dots, z_d)$. So we conclude that $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ produces at most a change of sign if we interchange \mathbf{u}_ρ with \mathbf{u}_τ . In particular, each u_{i0} appears effectively in F , and $\text{ord}(F, u_{i0})$ are the same for all $i = 0, 1, \dots, d$. Suppose $\text{ord}(F, u_{i0}) = s$. For $j \neq 0$, we consider $\text{ord}(F, u_{ij})$. If $\text{ord}(F, u_{ij}) = l > s$, then we differentiate $f(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$ on both sides w.r.t. $u_{ij}^{(l)}$. Thus $\frac{\partial f}{\partial u_{ij}^{(l)}}(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$, which amounts to a contradiction by Lemma 4.8. If $\text{ord}(F, u_{ij}) = l < s$, then we differentiate $f(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$ on both sides w.r.t. $u_{ij}^{(s)}$. Thus $\frac{\partial f}{\partial u_{i0}^{(s)}}(\mathbf{u}; \zeta_0, \dots, \zeta_d) \cdot (-\xi_j) = 0$. Since $\frac{\partial f}{\partial u_{i0}^{(s)}}(\mathbf{u}; \zeta_0, \dots, \zeta_d) \neq 0$, we have $\xi_j = 0$. Also, $y_j \in \mathbb{I}(V) \iff \xi_j = 0 \iff \zeta_i$ is free of $u_{ij} \iff \frac{\partial f}{\partial u_{ij}^{(k)}} = 0$ for all $k \in \mathbb{Z}^+ \iff u_{ij}$ does not appear in F . From the above, if u_{ij} occurs effectively in F , $\text{ord}(F, u_{ij}) = s$, which completes the theorem. \square

The *order of the Chow form* is defined to be $\text{ord}(f) = \text{ord}(F) = \text{ord}(f, u_{i0})$ for any $i \in \{0, \dots, d\}$. By Lemma 4.9, $\text{ord}(f)$ is equal to $\text{ord}(F, u_{ij})$ for those u_{ij} occurring in F .

The following lemma gives a property for the δ -ideal $\mathbb{I}_{\zeta, \xi}$ defined in Lemma 4.3.

Lemma 4.10. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, u_{10}, \dots, u_{d0})$ be the Chow form of a prime δ - \mathcal{F} -ideal \mathcal{I} and $s = \text{ord}(f)$. Then*

$$\mathcal{A} = \left\{ f, S_f y_1 - \frac{\partial f}{\partial u_{01}^{(s)}}, \dots, S_f y_n - \frac{\partial f}{\partial u_{0n}^{(s)}} \right\}$$

is a characteristic set of the prime δ -ideal $\mathbb{I}_{\zeta, \xi} = [\mathcal{I}, \mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_d]$ in $\mathcal{F}(\mathbf{u})\{u_{00}, u_{10}, \dots, u_{d0}, \mathbb{Y}\}$ w.r.t. the elimination ranking $u_{d0} \prec \dots \prec u_{00} \prec y_1 \prec \dots \prec y_n$, where $S_f = \frac{\partial f}{\partial u_{00}^{(s)}}$.

Proof. From Lemma 4.3, $\mathbb{I}_{\zeta, \xi}$ is a prime δ -ideal of dimension d with a generic point $(\zeta_0, \dots, \zeta_d, \xi_1, \dots, \xi_n)$. From Lemma 4.1, u_{10}, \dots, u_{d0} is a parametric set of $\mathbb{I}_{\zeta, \xi}$. If we differentiate $f(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$ w.r.t. $u_{0\rho}^{(s)}$ ($\rho = 1, \dots, n$), then we have $\frac{\partial f}{\partial u_{0\rho}^{(s)}} - \xi_\rho \bar{S}_f = 0$, where $\frac{\partial f}{\partial u_{0\rho}^{(s)}}$ and \bar{S}_f are obtained by replacing (u_{00}, \dots, u_{d0}) with $(\zeta_0, \dots, \zeta_d)$ in $\frac{\partial f}{\partial u_{0\rho}^{(s)}}$ and S_f , respectively. So $T_\rho = S_f y_\rho - \frac{\partial f}{\partial u_{0\rho}^{(s)}} \in \mathbb{I}_{\zeta, \xi}$. Since f is irreducible, we have $S_f \notin \mathbb{I}_{\zeta, \xi}$. Also note that T_i is linear in y_i . \mathcal{A} must be a characteristic set² of $\mathbb{I}_{\zeta, \xi}$ w.r.t. the elimination ranking $u_{d0} \prec \dots \prec u_{00} \prec y_1 \prec \dots \prec y_n$. \square

Now we give the first main property of the differential Chow form.

Theorem 4.11. *Let \mathcal{I} be a prime δ - \mathcal{F} -ideal with dimension d , and $f(\mathbf{u}; u_{00}, u_{10}, \dots, u_{d0})$ its Chow form. Then $\text{ord}(f) = \text{ord}(\mathcal{I})$.*

Proof. Use the notation ξ_i, ζ_i , and \mathbb{P}_i introduced in (4.2) and (4.5). Let $\mathcal{I}_d = [\mathcal{I}, \mathbb{P}_1, \dots, \mathbb{P}_d] \subset \mathcal{F}(\mathbf{u}_1, \dots, \mathbf{u}_d)\{\mathbb{Y}\}$. By Theorem 3.14, \mathcal{I}_d is a prime δ -ideal with dimension 0 and the same order as \mathcal{I} .

Let $\mathcal{I}_1 = [\mathcal{I}, \mathbb{P}_0, \dots, \mathbb{P}_d] = [\mathcal{I}_d, \mathbb{P}_0] \subset \mathcal{F}(\mathbf{u}_1, \dots, \mathbf{u}_d; u_{01}, \dots, u_{0n})\{u_{00}, y_1, \dots, y_n\}$. From Lemma 4.10, $\mathcal{A} = \left\{ f, S_f y_1 - \frac{\partial f}{\partial u_{01}^{(s)}}, \dots, S_f y_n - \frac{\partial f}{\partial u_{0n}^{(s)}} \right\}$ is a characteristic set

²Here \mathcal{A} is a δ -chain. See Remark 2.3.

of $\mathbb{I}_{\zeta, \xi}$. By Lemma 4.1, u_{10}, \dots, u_{d0} is a parametric set of $\mathbb{I}_{\zeta, \xi}$. So \mathcal{A} is also a characteristic set of \mathcal{I}_1 w.r.t. the elimination ranking $u_{00} \prec y_1 \prec \dots \prec y_n$. Since $\dim(\mathcal{I}_1) = 0$, from Corollary 2.10, we have $\text{ord}(\mathcal{I}_1) = \text{ord}(\mathcal{A}) = \text{ord}(f)$.

On the other hand, if (η_1, \dots, η_n) is a generic point of \mathcal{I}_d , then $(\eta_1, \dots, \eta_n, \zeta)$ is a generic point of \mathcal{I}_1 with $\zeta = -\sum_{j=1}^n u_{0j}\eta_j$ and $\dim(\mathcal{I}_1) = 0$. Clearly, u_{0k} ($k = 1, \dots, n$) are δ -independent over $\mathcal{F}\langle \mathbf{u}_1, \dots, \mathbf{u}_d, \eta_1, \dots, \eta_n \rangle$. Denote $\mathcal{F}\langle \mathbf{u}_1, \dots, \mathbf{u}_d, u_{01}, \dots, u_{0n} \rangle$ by \mathcal{F}_1 . So for a sufficiently large t ,

$$\begin{aligned} \omega_{\mathcal{I}_1}(t) &= \text{ord}(\mathcal{I}_1) \\ &= \text{tr.deg } \mathcal{F}_1(\eta_i^{(j)}, \zeta^{(j)} : i = 1, \dots, n; j \leq t) / \mathcal{F}_1 \\ &= \text{tr.deg } \mathcal{F}_1(\eta_i^{(j)} : i = 1, \dots, n; j \leq t) / \mathcal{F}_1 \\ &= \text{tr.deg } \mathcal{F}\langle \mathbf{u}_1, \dots, \mathbf{u}_d \rangle(\eta_i^{(j)} : i = 1, \dots, n; j \leq t) / \mathcal{F}\langle \mathbf{u}_1, \dots, \mathbf{u}_d \rangle \\ &= \omega_{\mathcal{I}_d}(t) = \text{ord}(\mathcal{I}_d). \end{aligned}$$

Thus, $\text{ord}(\mathcal{I}_1) = \text{ord}(\mathcal{I}_d) = \text{ord}(\mathcal{I})$, and consequently, $\text{ord}(\mathcal{I}) = \text{ord}(f)$. □

As a consequence, we can give an equivalent definition for the order of a prime δ -ideal using Chow forms.

Definition 4.12. Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension d and $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ its Chow form. The order of \mathcal{I} is defined to be the order of its Chow form.

The following result shows that we can recover the generic point (ξ_1, \dots, ξ_n) of V from its Chow form.

Theorem 4.13. Let $f(\mathbf{u}; u_{00}, \dots, u_{d0})$ be defined as above and h the order of V . Then we have

$$\xi_\rho = \frac{\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}}{\overline{S}_f}, \quad \rho = 1, \dots, n,$$

where $\frac{\overline{\partial f}}{\overline{\partial u_{0\rho}^{(h)}}}$ and \overline{S}_f are obtained by replacing (u_{00}, \dots, u_{d0}) by $(\zeta_0, \dots, \zeta_d)$ in $\frac{\partial f}{\partial u_{0\rho}^{(h)}}$ and $\frac{\partial f}{\partial u_{00}^{(h)}}$, respectively.

Proof. It follows directly from Lemma 4.10 and Theorem 4.11. □

Remark 4.14. The δ -resolvent [10] and [32, p.34] is closely related with the differential Chow form, although they are different. From Lemma 4.10 and Theorem 4.11, we can see that both of them can be used to reduce a δ -ideal \mathcal{I} to a δ -hypersurface which has the same order as \mathcal{I} in a certain sense. But they are quite different. First, the resolvent depends on a parameter set U of \mathcal{I} . To be more precise, let $Z = \mathbb{Y} \setminus U$. Then, the resolvent is essentially constructed in $\mathcal{F}\langle U \rangle\{Z\}$. Second, to define the resolvent, we need only one new indeterminate w and add one linear δ -polynomial $w - c_1v_1 - \dots - c_pv_p$ to \mathcal{I} , where $Z = \{v_1, \dots, v_p\}$. Furthermore, c_i in the above expression are from \mathcal{F} . Therefore, the resolvent will never be the Chow form. Of course, if \mathcal{I} is of dimension zero and we take c_i in $w - c_1y_1 - \dots - c_ny_n$ as δ -indeterminates, then the resolvent is the Chow form of \mathcal{I} . On the other hand, the resolvent of \mathcal{I} can be obtained from its Chow form by specializing some u_{ij} to certain specific values and using techniques in Theorem 2.16.

4.3. Differential Chow form is differentially homogenous. Following Kolchin [23], we introduce the concept of δ -homogenous δ -polynomials.

Definition 4.15. A δ -polynomial $p \in \mathcal{F}\{y_0, y_1, \dots, y_n\}$ is called δ -homogenous of degree m if for a new δ - \mathcal{F} -indeterminate λ , we have $p(\lambda y_0, \lambda y_1, \dots, \lambda y_n) = \lambda^m p(y_0, y_1, \dots, y_n)$.

The differential analog of Euler’s theorem related to homogenous polynomials is valid.

Theorem 4.16 ([19, p.71]). *A δ -polynomial $f \in \mathcal{F}\{y_0, y_1, \dots, y_n\}$ is δ -homogenous of degree m if and only if*

$$\sum_{j=0}^n \sum_{k \in \mathbb{N}} \binom{k+r}{r} y_j^{(k)} \frac{\partial f(y_0, \dots, y_n)}{\partial y_j^{(k+r)}} = \begin{cases} mf, & r = 0, \\ 0, & r \neq 0. \end{cases}$$

For the Chow form, we have the following result.

Theorem 4.17. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \dots, u_{d0})$ be the Chow form of an irreducible δ -variety V of dimension d and order h . Then*

1)

$$\sum_{j=0}^n u_{\tau j} \frac{\partial f}{\partial u_{\sigma j}} + \sum_{j=0}^n u'_{\tau j} \frac{\partial f}{\partial u'_{\sigma j}} + \dots + \sum_{j=0}^n u_{\tau j}^{(h)} \frac{\partial f}{\partial u_{\sigma j}^{(h)}} = \begin{cases} 0, & \sigma \neq \tau, \\ rf, & \sigma = \tau, \end{cases}$$

where r is a nonnegative integer.

2) $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ is a δ -homogenous δ -polynomial of degree r in each set \mathbf{u}_i and F is of total degree $(d + 1)r$.

Proof. Let (ξ_1, \dots, ξ_n) be a generic point of V and $\zeta_i = -\sum_{j=1}^n u_{ij} \xi_j$ ($i = 0, \dots, d$) defined in (4.2). From (4.3), $f(\mathbf{u}; u_{00}, \dots, u_{d0})$ is the characteristic set of the prime δ -ideal \mathbb{I}_ζ . Since $f(\mathbf{u}; \zeta_0, \zeta_1, \dots, \zeta_d) = 0$, we have

$$\frac{\partial f}{\partial u_{\sigma j}} + \frac{\partial f}{\partial \zeta_\sigma} (-\xi_j) + \frac{\partial f}{\partial \zeta'_\sigma} (-\xi'_j) + \frac{\partial f}{\partial \zeta''_\sigma} (-\binom{2}{1} \xi''_j) + \dots + \frac{\partial f}{\partial \zeta_\sigma^{(h)}} [-\binom{h}{0} \xi_j^{(h)}] = 0 \quad (0*)$$

$$\frac{\partial f}{\partial u'_{\sigma j}} + 0 + \frac{\partial f}{\partial \zeta'_\sigma} (-\xi_j) + \frac{\partial f}{\partial \zeta''_\sigma} (-\binom{2}{1} \xi'_j) + \dots + \frac{\partial f}{\partial \zeta_\sigma^{(h)}} [-\binom{h}{1} \xi_j^{(h-1)}] = 0 \quad (1*)$$

$$\frac{\partial f}{\partial u''_{\sigma j}} + 0 + 0 + \frac{\partial f}{\partial \zeta''_\sigma} (-\binom{2}{2} \xi_j) + \dots + \frac{\partial f}{\partial \zeta_\sigma^{(h)}} [-\binom{h}{2} \xi_j^{(h-2)}] = 0 \quad (2*)$$

.....

$$\frac{\partial f}{\partial u_{\sigma j}^{(h)}} + 0 + 0 + 0 + 0 + \frac{\partial f}{\partial \zeta_\sigma^{(h)}} [-\binom{h}{h} \xi_j^{(0)}] = 0 \quad (h*).$$

In the above equations, $\frac{\partial f}{\partial u_{\sigma j}^{(l)}}$ and $\frac{\partial f}{\partial \zeta_\sigma^{(l)}}$ ($l = 0, \dots, h; j = 1, \dots, n$) are obtained by substituting ζ_i to u_{i0} ($i = 0, 1, \dots, d$) in each $\frac{\partial f}{\partial u_{\sigma j}^{(l)}}$ and $\frac{\partial f}{\partial u_{\sigma 0}^{(l)}}$, respectively.

Now, let us consider the δ -polynomial $\sum_{j=0}^n \sum_{k \geq 0} \binom{k+i}{k} u_{\sigma j}^{(k)} \frac{\partial f}{\partial u_{\sigma j}^{(k+i)}}$.

In the case $i = 0$, first, let $(0*) \times u_{\tau j} + (1*) \times u'_{\tau j} + \dots + (h*) \times u_{\tau j}^{(h)}$ and add them together for j from 1 to n . We obtain

$$\sum_{j=1}^n u_{\tau j} \frac{\partial f}{\partial u_{\sigma j}} + \sum_{j=1}^n u'_{\tau j} \frac{\partial f}{\partial u'_{\sigma j}} + \dots + \sum_{j=1}^n u_{\tau j}^{(h)} \frac{\partial f}{\partial u_{\sigma j}^{(h)}} + \zeta_\tau \frac{\partial f}{\partial \zeta_\sigma} + \zeta'_\tau \frac{\partial f}{\partial \zeta'_\sigma} + \dots + \zeta_\tau^{(h)} \frac{\partial f}{\partial \zeta_\sigma^{(h)}} = 0.$$

So the δ -polynomial $\sum_{j=0}^n u_{\tau j} \frac{\partial f}{\partial u_{\sigma j}} + \sum_{j=0}^n u'_{\tau j} \frac{\partial f}{\partial u'_{\sigma j}} + \sum_{j=0}^n u''_{\tau j} \frac{\partial f}{\partial u''_{\sigma j}} + \dots + \sum_{j=0}^n u_{\tau j}^{(h)} \frac{\partial f}{\partial u_{\sigma j}^{(h)}}$ vanishes at $(u_{00}, \dots, u_{d0}) = (\zeta_0, \dots, \zeta_d)$. Thus in the case $\tau = \sigma$, it can be divisible

by f , i.e. $\sum_{j=0}^n \sum_{k=0}^h u_{\sigma j}^{(k)} \frac{\partial f}{\partial u_{\sigma j}^{(k)}} = rf$. By Euler's theorem, f is an algebraic homogenous δ -polynomial of degree r in each set of indeterminates $\mathbf{u}_i = (u_{i0}, \dots, u_{in})$ and their derivatives. But in the case $\tau \neq \sigma$, since this δ -polynomial is of order not greater than f and cannot be divisible by f , by Lemma 4.8, it must be identically zero. Thus, we have proved 1) of the Theorem.

In the case $i \neq 0$,

$$\begin{aligned} 0 &= (i^*) \times \binom{i}{i} u_{\sigma j} + (i+1^*) \times \binom{i+1}{i} u'_{\sigma j} + \dots + (h^*) \times \binom{h}{i} u_{\sigma j}^{(h-i)} \\ &= \binom{i}{i} u_{\sigma j} \frac{\partial f}{\partial u_{\sigma j}^{(i)}} + \binom{i+1}{i} u'_{\sigma j} \frac{\partial f}{\partial u_{\sigma j}^{(i+1)}} + \dots + \binom{h}{i} u_{\sigma j}^{(h-i)} \frac{\partial f}{\partial u_{\sigma j}^{(h)}} \\ &\quad + \frac{\partial f}{\partial \zeta_{\sigma}^{(i)}} \left(-\binom{i}{i} u_{\sigma j} \xi_j \right) + \frac{\partial f}{\partial \zeta_{\sigma}^{(i+1)}} \left(-\binom{i+1}{i} u_{\sigma j} \xi'_j - \binom{i+1}{i} \binom{i+1}{i+1} u'_{\sigma j} \xi_j \right) \\ &\quad + \dots \\ &\quad + \frac{\partial f}{\partial \zeta_{\sigma}^{(h)}} \left(-\binom{i}{i} \binom{h}{i} u_{\sigma j} \xi_j^{(h-i)} - \binom{i+1}{i} \binom{h}{i+1} u'_{\sigma j} \xi_j^{(h-i-1)} - \dots - \binom{h}{i} \binom{h}{i} u_{\sigma j}^{(h-i)} \xi_j \right) \\ &= \binom{i}{i} u_{\sigma j} \frac{\partial f}{\partial u_{\sigma j}^{(i)}} + \binom{i+1}{i} u'_{\sigma j} \frac{\partial f}{\partial u_{\sigma j}^{(i+1)}} + \dots + \binom{h}{i} u_{\sigma j}^{(h-i)} \frac{\partial f}{\partial u_{\sigma j}^{(h)}} \\ &\quad + \binom{i}{i} \frac{\partial f}{\partial \zeta_{\sigma}^{(i)}} (-u_{\sigma j} \xi_j) + \binom{i+1}{i} \frac{\partial f}{\partial \zeta_{\sigma}^{(i+1)}} (-u_{\sigma j} \xi'_j) + \dots + \binom{h}{i} \frac{\partial f}{\partial \zeta_{\sigma}^{(h)}} (-u_{\sigma j} \xi_j)^{(h-i)}. \end{aligned}$$

Therefore, $\sum_{j=1}^n \binom{i}{i} u_{\sigma j} \frac{\partial f}{\partial u_{\sigma j}^{(i)}} + \sum_{j=1}^n \binom{i+1}{i} u'_{\sigma j} \frac{\partial f}{\partial u_{\sigma j}^{(i+1)}} + \dots + \sum_{j=1}^n \binom{h}{i} u_{\sigma j}^{(h-i)} \frac{\partial f}{\partial u_{\sigma j}^{(h)}} + \binom{i}{i} \zeta_{\sigma} \frac{\partial f}{\partial \zeta_{\sigma}^{(i)}} + \binom{i+1}{i} \zeta'_{\sigma} \frac{\partial f}{\partial \zeta_{\sigma}^{(i+1)}} + \dots + \binom{h}{i} \zeta_{\sigma}^{(h-i)} \frac{\partial f}{\partial \zeta_{\sigma}^{(h)}} = 0$.

Thus, the δ -polynomial $\sum_{j=0}^n \binom{i}{i} u_{\sigma j} \frac{\partial f}{\partial u_{\sigma j}^{(i)}} + \sum_{j=0}^n \binom{i+1}{i} u'_{\sigma j} \frac{\partial f}{\partial u_{\sigma j}^{(i+1)}} + \dots + \sum_{j=0}^n \binom{h}{i} u_{\sigma j}^{(h-i)} \frac{\partial f}{\partial u_{\sigma j}^{(h)}}$ is identically zero, for it vanishes at $(u_{00}, \dots, u_{d0}) = (\zeta_0, \dots, \zeta_d)$ and cannot be divisible by f .

From the above, we conclude that

$$\sum_{j=0}^n \sum_{k \geq 0} \binom{k+i}{i} u_{\sigma j}^{(k)} \frac{\partial f}{\partial u_{\sigma j}^{(k+i)}} = \begin{cases} 0, & i \neq 0, \\ rf, & i = 0. \end{cases}$$

From Theorem 4.16 and the symmetry property of $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$, the theorem is obtained. □

Lemma 4.9, Theorem 4.11, and Theorem 4.17 together prove the first property of Theorem 1.2.

Remark 4.18. Using the terminology from [24], the differential Chow form $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ is $(d+1)$ - δ -homogenous in $(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$.

Definition 4.19. Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ of dimension d and $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ its Chow form. The δ -degree of \mathcal{I} is defined to be the δ -homogenous degree r of its Chow form in each \mathbf{u}_i ($i = 0, \dots, d$).

The following result shows that the δ -degree of a δ -variety V is an invariant of V under invertible linear transformations.

Lemma 4.20. *Let $A = (a_{ij})$ be an $n \times n$ invertible matrix with $a_{ij} \in \mathcal{F}$ and $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ be the Chow form of an irreducible δ -variety V with dimension d . Then the Chow form of the image δ -variety of V under the linear transformation*

$\mathbb{Y} = A\mathbb{X}$ is $F^A(\mathbf{v}_0, \dots, \mathbf{v}_d) = F(\mathbf{v}_0B, \dots, \mathbf{v}_dB)$, where $B = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \cdots 0 \\ & & & A \end{pmatrix}$ and \mathbf{u}_i and \mathbf{v}_i are regarded as row vectors.

Proof. Let $\xi = (\xi_1, \dots, \xi_n)$ be a generic point of V . Under the linear transformation $\mathbb{Y} = A\mathbb{X}$, ξ is mapped to $\eta = (\eta_1, \dots, \eta_n)$ with $\eta_i = \sum_{j=1}^n a_{ij}\xi_j$. Under this transformation V is mapped to a δ -variety V^A whose generic point is η . Note that $F^A(\mathbf{v}_0, \dots, \mathbf{v}_d) = f^A(v_{ij}; v_{00}, \dots, v_{d0}) = f(\sum_{k=1}^n v_{ik}a_{kj}; v_{00}, \dots, v_{d0})$ and $f^A(v_{ij}; -\sum_{k=1}^n v_{0k}\eta_k, \dots, -\sum_{k=1}^n v_{dk}\eta_k) = f(\sum_{k=1}^n v_{ik}a_{kj}; -\sum_{k=1}^n v_{0k}\eta_k, \dots, -\sum_{k=1}^n v_{dk}\eta_k) = f(\sum_{k=1}^n v_{ik}a_{kj}; -\sum_{j=1}^n (\sum_{k=1}^n v_{0k}a_{kj})\xi_j, \dots, -\sum_{j=1}^n (\sum_{k=1}^n v_{dk} a_{kj})\xi_j) = 0$. Since V^A is of the same dimension and order as V and F^A is irreducible, from the definition of the Chow form, the claim is proved. \square

Definition 4.21. Let p be a δ -polynomial in $\mathcal{F}\{\mathbb{Y}\}$. Then the smallest number r such that $y_0^r p(y_1/y_0, \dots, y_n/y_0) \in \mathcal{F}\{y_0, y_1, \dots, y_n\}$ is called the denominator of p , which is denoted by $\text{den}(p)$.

Example 4.22. In the case $d = n - 1$ and $n > 1$, if $\{p(y_1, \dots, y_n)\}$ is a characteristic set of \mathcal{I} w.r.t. any ranking, then by Example 4.6 the δ -degree of \mathcal{I} cannot exceed the denominator of p . So the denominator of p gives an upper bound of the δ -degree of \mathcal{I} . But, we do not know whether they are the same.

Example 4.23. In the case $n = 1$, we have $d = 0$. If $\{p(y)\}$ is a characteristic set of \mathcal{I} w.r.t. any ranking, then the δ -degree of \mathcal{I} is exactly equal to the denominator of p . Now we can give a bound for the δ -degree of \mathcal{I} from the original equation of p without computing its denominator.

For a δ -monomial $\theta(y) = y^{l_0}(y')^{l_1}(y'')^{l_2} \dots (y^{(s)})^{l_s}$, define its *weighted degree* to be $l_0 + 2l_1 + \dots + (s + 1)l_s$, denoted by $\text{wdeg}(\theta(y)) = l_0 + 2l_1 + \dots + (s + 1)l_s$. For a δ -polynomial $p \in \mathcal{F}\{y\}$, we can define its weighted degree to be the maximum of the weighted degrees of all the δ -monomials effectively appearing in p . Clearly, the denominator of p cannot exceed its weighted degree, and we have examples for which $\text{den}(p) < \text{wdeg}(p)$. Let $p = 2y'^2 - yy''$. Then, $\text{den}(p) = 3$ and $\text{wdeg}(p) = 4$. The Chow form of $\text{sat}(p)$ is $F(\mathbf{u}) = u_0u_1u_0'' - u_0^2u_1'' - 2u_0''u_1 + 2u_0u_0'u_1'$, where $\mathbf{u} = (u_0, u_1)$. So the δ -degree of $\mathcal{I} = \text{sat}(p)$ is 3, which is less than $\text{wdeg}(p)$.

Now we contrast the above p with a δ -polynomial $q = y'^2 - yy''$ that is different from p by only a single coefficient. Then $\text{den}(q) = 4$ and $\text{wdeg}(q) = 4$. The Chow form of $\text{sat}(q)$ is $G(\mathbf{u}_0) = u_0u_1^2u_0'' - u_0^2u_1u_1'' - u_0''u_1^2 + u_0^2u_1'^2$, so the δ -degree of $\text{sat}(q)$ is 4, which is equal to the weighted degree of q . Thus, the weighted degree is a sharp bound.

4.4. Factorization of the differential Chow form. In the algebraic case, the Chow form can be factored into the product of linear polynomials with the generic points of the variety as coefficients. In this section, we will show that there is a differential analog to this result.

Let

$$\tilde{\mathbf{u}} = \bigcup_{i=0}^d \mathbf{u}_i \setminus \{u_{00}\}$$

and $\mathcal{F}_0 = \mathcal{F}(\tilde{\mathbf{u}})(u_{00}^{(0)}, \dots, u_{00}^{(h-1)})$. Consider the Chow form $f(\mathbf{u}; u_{00}, u_{10}, \dots, u_{d0})$ as an irreducible algebraic polynomial $p(u_{00}^{(h)})$ in $\mathcal{F}_0[u_{00}^{(h)}]$. Let $g = \text{deg}(p, u_{00}^{(h)}) =$

$\deg(f, u_{00}^{(h)})$. In a suitable algebraic extension field of \mathcal{F}_0 , $p(u_{00}^{(h)}) = 0$ has g roots $\gamma_1, \dots, \gamma_g$. Thus

$$(4.6) \quad f(\mathbf{u}; u_{00}, u_{10}, \dots, u_{d0}) = A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) \prod_{\tau=1}^g (u_{00}^{(h)} - \gamma_\tau),$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ is in $\mathcal{F}\{\mathbf{u}_0, \dots, \mathbf{u}_d\}$. For each τ such that $1 \leq \tau \leq g$, let

$$(4.7) \quad \mathcal{F}_\tau = \mathcal{F}_0(\gamma_\tau) = \mathcal{F}\langle \tilde{\mathbf{u}} \rangle(u_{00}^{(0)}, \dots, u_{00}^{(h-1)}, \gamma_\tau)$$

be an algebraic extension of \mathcal{F}_0 defined by $p(u_{00}^{(h)}) = 0$. We will define a derivation δ_τ on \mathcal{F}_τ so that $(\mathcal{F}_\tau, \delta_\tau)$ becomes a differential field. This can be done in a very natural way. For $e \in \mathcal{F}\langle \tilde{\mathbf{u}} \rangle$, define $\delta_\tau e = \delta e = e'$. Define $\delta_\tau^i u_{00} = u_{00}^{(i)}$ for $i = 0, \dots, h - 1$ and

$$\delta_\tau^h u_{00} = \gamma_\tau.$$

Since f , regarded as an algebraic polynomial p in $u_{00}^{(h)}$, is a minimal polynomial of γ_τ , $S_f = \frac{\partial f}{\partial u_{00}^{(h)}}$ does not vanish at $u_{00}^{(h)} = \gamma_\tau$. Now, we define the derivatives of $\delta_\tau^i u_{00}$ for $i > h$ by induction. First, since $p(\gamma_\tau) = 0$, $\delta_\tau(p(\gamma_\tau)) = S_f|_{u_{00}^{(h)}=\gamma_\tau} \delta_\tau(\gamma_\tau) + T|_{u_{00}^{(h)}=\gamma_\tau} = 0$, where $T = f' - S_f u_{00}^{(h+1)}$. We define $\delta_\tau^{h+1} u_{00}$ to be $\delta_\tau(\gamma_\tau) = -\frac{T}{S_f}|_{u_{00}^{(h)}=\gamma_\tau}$. Supposing the derivatives of $\delta_\tau^j u_{00}$ with order less than $j < i$ have been defined, we now define $\delta_\tau^i u_{00}$. Since $f^{(i)} = S_f u_{00}^{(h+i)} + T_i$ is linear in $u_{00}^{(h+i)}$, we define $\delta_\tau^i u_{00}$ to be $-\frac{T_i}{S_f}|_{u_{00}^{(h+j)}=\delta_\tau^{h+j} u_{00}, j < i}$.

In this way, $(\mathcal{F}_\tau, \delta_\tau)$ is a differential field which can be considered as a finitely differential extension field of $(\mathcal{F}\langle \tilde{\mathbf{u}} \rangle, \delta)$. Recall that $\mathcal{F}\langle \tilde{\mathbf{u}} \rangle$ is a finitely δ -extension field of \mathcal{F} contained in \mathcal{E} . By the definition of a universal δ -extension field, there exists a δ -extension field $\mathcal{F}^* \subset \mathcal{E}$ of $\mathcal{F}\langle \tilde{\mathbf{u}} \rangle$ and a δ -isomorphism φ_τ over $\mathcal{F}\langle \tilde{\mathbf{u}} \rangle$ from $(\mathcal{F}_\tau, \delta_\tau)$ to (\mathcal{F}^*, δ) . Summing up the above results, we have

Lemma 4.24. *$(\mathcal{F}_\tau, \delta_\tau)$ defined above is a finitely differential extension field of $\mathcal{F}\langle \tilde{\mathbf{u}} \rangle$, which is δ - $\mathcal{F}\langle \tilde{\mathbf{u}} \rangle$ -isomorphic to a subfield of \mathcal{E} .*

Let p be a δ -polynomial in $\mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d\} = \mathcal{F}\{\tilde{\mathbf{u}}, u_{00}\}$. For convenience, by the symbol $p|_{u_{00}^{(h)}=\gamma_\tau}$ or, say, replacing $u_{00}^{(h)}$ by γ_τ , we mean substituting $u_{00}^{(h+i)}$ by $\delta_\tau^i \gamma_\tau$ ($i \geq 0$) in p . Similarly, by saying p vanishes at $u_{00}^{(h)} = \gamma_\tau$, we mean $p|_{u_{00}^{(h)}=\gamma_\tau} = 0$. It is easy to prove the following lemma.

Lemma 4.25. *Let p be a δ -polynomial in $\mathcal{F}\{\tilde{\mathbf{u}}, u_{00}\}$ and f the Chow form for a prime δ - \mathcal{F} -ideal \mathcal{I} of dimension d . Then $p \in \text{sat}(f)$ if and only if p vanishes at $u_{00}^{(h)} = \gamma_\tau$.*

When a δ -polynomial $h \in \mathcal{F}\langle \tilde{\mathbf{u}} \rangle\{\mathbb{Y}\}$ vanishes at a point $\eta \in \mathcal{F}_\tau^n$, it is easy to see that h vanishes at $\varphi_\tau(\eta) \in \mathcal{E}^n$. For convenience, by saying η is in a δ -variety V over $\mathcal{F}\langle \tilde{\mathbf{u}} \rangle$, we mean $\varphi_\tau(\eta) \in V$.

Remark 4.26. In order to make \mathcal{F}_τ a differential field, we need to introduce a differential operator δ_τ which is related to γ_τ , and there does not exist a single differential operator to make all \mathcal{F}_τ ($\tau = 1, \dots, g$) differential fields. This natural

phenomenon related with nonlinear differential equations seems not to have been used before. For instance, let $p = y'^2 - 4y$. Then $\mathcal{I} = \text{sat}(p)$ is a prime δ -ideal in $\mathbb{Q}(t)\{y\}$ and let $\overline{\mathcal{F}}$ be the differential rational function field of \mathcal{I} . By factoring $p = (y' - 2\sqrt{y})(y' + 2\sqrt{y})$, we can define two more differential fields: $\mathcal{F}_1 = \mathbb{Q}(t)(\sqrt{y})$ with a differential operator $\delta_1 y = 2\sqrt{y}$ and $\mathcal{F}_2 = \mathbb{Q}(t)(\sqrt{y})$ with a differential operator $\delta_2 y = -2\sqrt{y}$. Note that \mathcal{F}_1 and \mathcal{F}_2 are not compatible, although each of them is isomorphic to a subfield of \mathcal{E} . Finally, both \mathcal{F}_1 and \mathcal{F}_2 are isomorphic to $\overline{\mathcal{F}}$.

With these preparations, we now give the following theorem.

Theorem 4.27. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \dots, u_{d0})$ be the Chow form of an irreducible δ - \mathcal{F} -variety of dimension d and order h . Then, there exist $\xi_{\tau 1}, \dots, \xi_{\tau n}$ in a differential extension field $(\mathcal{F}_\tau, \delta_\tau)$ ($\tau = 1, \dots, g$) of $(\mathcal{F}(\tilde{\mathbf{u}}), \delta)$ such that*

$$(4.8) \quad F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) \prod_{\tau=1}^g (u_{00} + \sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})^{(h)},$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ is in $\mathcal{F}\{\mathbf{u}_0, \dots, \mathbf{u}_d\}$, $\tilde{\mathbf{u}} = \bigcup_{i=0}^d \mathbf{u}_i \setminus u_{00}$ and $g = \text{deg}(f, u_{00}^{(h)})$. Note that equation (4.8) is formal and should be understood in the following precise meaning: $(u_{00} + \sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})^{(h)} \triangleq \delta^h u_{00} + \delta_\tau^h (\sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})$.

Proof. We will follow the notation introduced in the proof of Lemma 4.24. Since f is irreducible, we have $f_{\tau 0} = \frac{\partial f}{\partial u_{00}^{(h)}} \Big|_{u_{00}^{(h)} = \gamma_\tau} \neq 0$. Let $\xi_{\tau\rho} = f_{\tau\rho} / f_{\tau 0}$ ($\rho = 1, \dots, n$), where $f_{\tau\rho} = \frac{\partial f}{\partial u_{0\rho}^{(h)}} \Big|_{u_{00}^{(h)} = \gamma_\tau}$. Note that $f_{\tau\rho}$ and $\xi_{\tau\rho}$ are in \mathcal{F}_τ . We will prove

$$\gamma_\tau = -\delta_\tau^h (u_{01} \xi_{\tau 1} + u_{02} \xi_{\tau 2} + \dots + u_{0n} \xi_{\tau n}).$$

Differentiating the equality $f(\mathbf{u}; \zeta_0, \zeta_1, \dots, \zeta_d) = 0$ w.r.t. $u_{0\rho}^{(h)}$, we have

$$\frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}} + \frac{\overline{\partial f}}{\partial u_{00}^{(h)}} (-\xi_\rho) = 0,$$

where $\frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}}$ are obtained by substituting ζ_i to u_{i0} ($i = 0, 1, \dots, d$) in $\frac{\partial f}{\partial u_{0\rho}^{(h)}}$. Multiplying $u_{0\rho}$ to the above equation and for ρ from 1 to n , adding them together, we have

$$\sum_{\rho=1}^n u_{0\rho} \frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}} + \frac{\overline{\partial f}}{\partial u_{00}^{(h)}} (-\sum_{\rho=1}^n u_{0\rho} \xi_\rho) = \sum_{\rho=1}^n u_{0\rho} \frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}} + \zeta_0 \frac{\overline{\partial f}}{\partial u_{00}^{(h)}} = 0.$$

Thus, $q = \sum_{\rho=1}^n u_{0\rho} \frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}} + u_{00} \frac{\overline{\partial f}}{\partial u_{00}^{(h)}} \in \text{sat}(f)$. Since q is of order not greater than f , it must be divisible by f . Since q and f have the same degree, there exists an $a \in \mathcal{F}$ such that $q = af$. Setting $u_{00}^{(h)} = \gamma_\tau$ on both sides of $q = af$, we have $\sum_{\rho=1}^n u_{0\rho} f_{\tau\rho} + u_{00} f_{\tau 0} = 0$. Hence, as an algebraic equation, we have

$$(4.9) \quad u_{00} + \sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho} = 0$$

under the constraint $u_{00}^{(h)} = \gamma_\tau$. Equivalently, the above equation is valid in $(\mathcal{F}_\tau, \delta_\tau)$. As a consequence, $\gamma_\tau = -\delta_\tau^h (\sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})$. Substituting them into equation (4.6), the theorem is proved. □

Remark 4.28. The factors in equation (4.8) are the h -th derivatives of the factors in the corresponding factorization of the algebraic Chow form [16, p.37].

For an element $\eta = (\eta_1, \dots, \eta_n)$, denote its truncation up to order k as $\eta^{[k]} = (\eta_1, \dots, \eta_m, \dots, \eta_1^{(k)}, \dots, \eta_m^{(k)})$.

In the proof of Theorem 4.27, some equations are valid in the algebraic case only. To avoid confusion, we introduce the following notation:

$$\begin{aligned}
 (4.10) \quad & a\mathbb{P}_0^{(0)} = a\mathbb{P}_0 := u_{00} + u_{01}y_1 + \dots + u_{0n}y_n, \\
 & a\mathbb{P}_0^{(1)} = a\mathbb{P}'_0 := u'_{00} + u'_{01}y_1 + u_{01}y'_1 + \dots + u'_{0n}y_n + u_{0n}y'_n \\
 & \dots \\
 & a\mathbb{P}_0^{(s)} := u_{00}^{(s)} + \sum_{j=1}^n \sum_{k=0}^s \binom{s}{k} u_{0j}^{(k)} y_j^{(s-k)},
 \end{aligned}$$

which are considered to be algebraic polynomials in $\mathcal{F}(\mathbf{u}_0^{[s]}, \dots, \mathbf{u}_n^{[s]})[\mathbb{Y}^{[s]}]$, and $u_{ij}^{(k)}, y_i^{(j)}$ are treated as algebraic indeterminates. A point $\eta = (\eta_1, \dots, \eta_n)$ is said to be lying on $a\mathbb{P}_0^{(k)}$ if regarded as an algebraic point, $\eta^{[k]}$ is a zero of $a\mathbb{P}_0^{(k)}$. As a consequence of (4.9) in the proof of Theorem 4.27, we have

Corollary 4.29. $(\xi_{\tau_1}, \dots, \xi_{\tau n}, \dots, \delta_\tau^{(h-1)}\xi_{\tau 1}, \dots, \delta_\tau^{(h-1)}\xi_{\tau n})$ ($\tau = 1, \dots, g$) are common zeros of $a\mathbb{P}_0 = 0, a\mathbb{P}'_0 = 0, \dots, a\mathbb{P}_0^{(h-1)} = 0$ where δ_τ is defined in Lemma 4.24.

Example 4.30. Continue from Example 4.5. In this example, $F(\mathbf{u}) = u_{01}^2(u'_{00})^2 - 2u_{01}u'_{01}u_{00}u'_{00} + (u'_{01})^2u_{00}^2 + 4u_{01}^3u_{00}$, so $g = 2$. Also, $F(\mathbf{u}) = u_{01}^2(u'_{00} - u'_{01}\frac{u_{00}}{u_{01}} + 2\sqrt{-1}\sqrt{u_{00}u_{01}})(u'_{00} - u'_{01}\frac{u_{00}}{u_{01}} - 2\sqrt{-1}\sqrt{u_{00}u_{01}})$. So we can obtain $\gamma_\tau = u'_{01}\frac{u_{00}}{u_{01}} \mp 2\sqrt{-1}\sqrt{u_{00}u_{01}}$ ($\tau = 1, 2$). Following the proof of Theorem 4.27, we obtain that $\xi_{11} = -u_{00}/u_{01}$, with u_{00}, u_{01} satisfying the relation $\delta_1 u_{00} = \gamma_1 = \frac{u_{00}}{u_{01}}u'_{01} - 2\sqrt{-1}\sqrt{u_{00}u_{01}}$, and $\xi_{21} = -u_{00}/u_{01}$, with u_{00}, u_{01} satisfying $\delta_2 u_{00} = \gamma_2 = \frac{u_{00}}{u_{01}}u'_{01} + 2\sqrt{-1}\sqrt{u_{00}u_{01}}$. In other words, ξ_{11} and ξ_{21} are in \mathcal{F}_1 and \mathcal{F}_2 , respectively. Thus, $F(\mathbf{u})$ has the factorization $F(\mathbf{u}) = u_{01}^2(u'_{00} + \xi_{11}u'_{01} + 2\sqrt{-1}\sqrt{u_{00}u_{01}})(u'_{00} + \xi_{21}u'_{01} - 2\sqrt{-1}\sqrt{u_{00}u_{01}}) = u_{01}^2(u_{00} + \xi_{11}u_{01})'(u_{00} + \xi_{21}u_{01})'$. Note that both ξ_{i1} ($i = 1, 2$) satisfy $a\mathbb{P}_0 = u_{00} + u_{01}\xi_{i1} = 0$, but $a\mathbb{P}_0^{(1)} = u'_{00} + u'_{01}\xi_{i1} + u_{01}\delta_i\xi_{i1} \neq 0$.

Lemma 4.31. In equation (4.8), $A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ is free of $u_{0i}^{(h)}$ ($i = 1, \dots, n$).

Proof. Since f is homogenous in the indeterminates u_{0i} and its derivatives up to the order h , we have

$$\sum_{\rho=1}^n u_{0\rho}^{(h)} \frac{\partial f}{\partial u_{0\rho}^{(h)}} + u_{00}^{(h)} \frac{\partial f}{\partial u_{00}^{(h)}} + \sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}} = r f, \quad r \in \mathbb{N}.$$

In this equation, let $u_{00}^{(h)} = \gamma_\tau$. We obtain

$$\sum_{\rho=1}^n u_{0\rho}^{(h)} f_{\tau\rho} + \gamma_\tau f_{\tau 0} + \sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \widehat{\frac{\partial f}{\partial u_{0\rho}^{(k)}}} = 0,$$

where $\widehat{\frac{\partial f}{\partial u_{0\rho}^{(k)}}}$ means replacing $u_{00}^{(h)}$ by γ_τ in $\frac{\partial f}{\partial u_{0\rho}^{(k)}}$. Consequently,

$$\gamma_\tau = - \sum_{\rho=1}^n u_{0\rho}^{(h)} \xi_{\tau\rho} - \sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \widehat{\frac{\partial f}{\partial u_{0\rho}^{(k)}}} / f_{\tau 0}.$$

Hence,
(4.11)

$$f(\mathbf{u}; u_{00}, u_{10}, \dots, u_{d0}) = A \prod_{\tau=1}^g \left(u_{00}^{(h)} + \sum_{\rho=1}^n u_{0\rho}^{(h)} \xi_{\tau\rho} + \sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\widehat{\partial f}}{\partial u_{0\rho}^{(k)}} / f_{\tau 0} \right).$$

We claim that $\xi_{\tau\rho}$ and $\sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\widehat{\partial f}}{\partial u_{0\rho}^{(k)}} / f_{\tau 0}$ are algebraically independent of $u_{0i}^{(h)}$ ($i = 1, \dots, n$). First, since ξ_{ρ} is algebraically independent of $u_{0i}^{(h)}$ ($i = 1, \dots, n$), and by Theorem 4.13,

$$\xi_{\rho} = \frac{\partial f}{\partial u_{0\rho}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}} \Big|_{(u_{00}, \dots, u_{d0}) = (\zeta_0, \dots, \zeta_d)} = \frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}} / \frac{\overline{\partial f}}{\partial u_{00}^{(h)}},$$

we have

$$\frac{\partial \xi_{\rho}}{\partial u_{0i}^{(h)}} = \frac{\frac{\partial}{\partial u_{0i}^{(h)}} \left(\frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}} \right) \frac{\overline{\partial f}}{\partial u_{00}^{(h)}} - \frac{\partial}{\partial u_{0i}^{(h)}} \left(\frac{\overline{\partial f}}{\partial u_{00}^{(h)}} \right) \frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}}}{\left(\frac{\overline{\partial f}}{\partial u_{00}^{(h)}} \right)^2} = 0,$$

where $\frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}}$ is obtained by replacing (u_{00}, \dots, u_{d0}) by $(\zeta_0, \dots, \zeta_d)$ in $\frac{\partial f}{\partial u_{0\rho}^{(h)}}$, or equivalently,

$$\frac{\partial}{\partial u_{0i}^{(h)}} \left(\frac{\partial f}{\partial u_{0\rho}^{(h)}} \right) \frac{\partial f}{\partial u_{00}^{(h)}} - \frac{\partial}{\partial u_{0i}^{(h)}} \left(\frac{\partial f}{\partial u_{00}^{(h)}} \right) \frac{\partial f}{\partial u_{0\rho}^{(h)}} \in \text{sat}(f).$$

Set $u_{00}^{(h)} = \gamma_{\tau}$; we have

$$\frac{\partial}{\partial u_{0i}^{(h)}} \left(f_{\tau\rho} \right) f_{\tau 0} - \frac{\partial}{\partial u_{0i}^{(h)}} \left(f_{\tau 0} \right) f_{\tau\rho} = 0.$$

Thus,

$$(4.12) \quad \frac{\partial \xi_{\tau\rho}}{\partial u_{0i}^{(h)}} = \frac{\partial (f_{\tau\rho} / f_{\tau 0})}{\partial u_{0i}^{(h)}} = 0.$$

Second, set $(u_{00}, \dots, u_{d0}) = (\zeta_0, \dots, \zeta_d)$ in the equation

$$\sum_{\rho=1}^n u_{0\rho}^{(h)} \frac{\partial f}{\partial u_{0\rho}^{(h)}} + u_{00}^{(h)} \frac{\partial f}{\partial u_{00}^{(h)}} + \sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}} = r f, \quad r \in \mathbb{N}.$$

We have

$$\sum_{\rho=1}^n u_{0\rho}^{(h)} \frac{\overline{\partial f}}{\partial u_{0\rho}^{(h)}} + \zeta_0^{(h)} \frac{\overline{\partial f}}{\partial u_{00}^{(h)}} + \sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\overline{\partial f}}{\partial u_{0\rho}^{(k)}} = 0.$$

By Theorem 4.13,

$$\left[\sum_{\rho=1}^n u_{0\rho}^{(h)} \xi_{\rho} + \zeta_0^{(h)} + \sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\overline{\partial f}}{\partial u_{0\rho}^{(k)}} \right] / \sqrt{\frac{\overline{\partial f}}{\partial u_{00}^{(h)}}} = 0.$$

Then,

$$\frac{\partial}{\partial u_{0i}^{(h)}} \left(\overline{\sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}}} / \overline{\frac{\partial f}{\partial u_{00}^{(h)}}} \right) = -\xi_i - (-\xi_i) = 0$$

$$= \frac{\frac{\partial}{\partial u_{0i}^{(h)}} \left(\overline{\sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}}} \right) \overline{\frac{\partial f}{\partial u_{00}^{(h)}}} - \frac{\partial}{\partial u_{0i}^{(h)}} \left(\overline{\frac{\partial f}{\partial u_{00}^{(h)}}} \right) \left(\overline{\sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}}} \right)}{\left(\overline{\frac{\partial f}{\partial u_{00}^{(h)}}} \right)^2}.$$

Thus, we have $\frac{\partial}{\partial u_{0i}^{(h)}} \left(\overline{\sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}}} \right) \overline{\frac{\partial f}{\partial u_{00}^{(h)}}} - \frac{\partial}{\partial u_{0i}^{(h)}} \left(\overline{\frac{\partial f}{\partial u_{00}^{(h)}}} \right) \left(\overline{\sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}}} \right) \in \widehat{\text{sat}}(f)$. From Lemma 4.25, by setting $u_{00}^{(h)} = \gamma_\tau$ in the above relation, we obtain

$$(4.13) \quad \frac{\partial \left(\overline{\sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}}} / f_{\tau 0} \right)}{\partial u_{0i}^{(h)}} = 0.$$

From (4.12) and (4.13), $\xi_{\tau\rho}$ and $\sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}} / f_{\tau 0}$ are algebraically independent of $u_{0i}^{(h)}$ ($i = 1, \dots, n$). Then the symmetric functions of $\xi_{\tau 1}, \dots, \xi_{\tau n}, \sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}} / f_{\tau 0}$ are rational functions in the set of indeterminates $\{u_{ik}, \dots, u_{ik}^{(h)}, u_{0k}, \dots, u_{0k}^{(h-1)} : i = 1, \dots, d; k = 0, \dots, n\}$ only. Therefore, $\prod_{\tau=1}^g (u_{00}^{(h)} - \gamma_\tau) = \frac{\phi}{\psi}$ where ψ is free of $u_{0i}^{(h)}$ ($i = 1, \dots, n$) and $\text{gcd}(\phi, \psi) = 1$. Thus $A\phi = f\psi$. Since f is irreducible, we conclude that $A = \psi$ is free of $u_{0i}^{(h)}$ ($i = 1, \dots, n$). \square

Note that the factorization (4.8) is formal in the sense that different factors are from different differential fields \mathcal{F}_τ . The following result shows how to obtain a factorization in the same extension field.

Theorem 4.32. *The quantities $\xi_{\tau 1}, \dots, \xi_{\tau n}$ in (4.8) are unique and (4.11) is a factorization of F as an algebraic polynomial in $u_{00}^{(h)}, \dots, u_{0n}^{(h)}$ in an extension field of $\mathcal{F}(u_{ik}, \dots, u_{ik}^{(h)}, u_{0k}, \dots, u_{0k}^{(h-1)} : i = 1, \dots, d; k = 0, \dots, n)$.*

Proof. From Lemma 4.31, equations (4.12) and (4.13), we can see that $A(\mathbf{u}_0, \dots, \mathbf{u}_d)$, $\xi_{\tau j}$, and $\sum_{k=0}^{h-1} \sum_{\rho=0}^n u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}} / f_{\tau 0}$ are free of $u_{0i}^{(h)}$ ($i = 1, \dots, n$). Then, (4.11) is a factorization of the Chow form $F(\mathbf{u}_0, \dots, \mathbf{u}_n)$ in the polynomial ring $\mathcal{F}(u_{ik}, \dots, u_{ik}^{(h)}, u_{0k}, \dots, u_{0k}^{(h-1)} : i = 1, \dots, d; k = 0, \dots, n)[u_{00}^{(h)}, \dots, u_{0n}^{(h)}]$. Thus, the factorization (4.11) must be unique, and hence $\xi_{\tau i}$. \square

4.5. Leading differential degree of an irreducible differential variety. In this subsection, we will define the leading differential degree for a prime δ -ideal and give its geometric meaning.

Definition 4.33. Let $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \dots, u_{d0})$ be the Chow form of a prime δ - \mathcal{F} -ideal \mathcal{I} of dimension d and order h . By Lemma 4.20, the number $g = \text{deg}(f, u_{00}^{(h)})$ is an invariant of \mathcal{I} under invertible linear transformations, which is called the *leading differential degree* of \mathcal{I} .

From (4.8), there exist g points $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ ($\tau = 1, \dots, g$), which have interesting properties.

Theorem 4.34. *The points $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ ($\tau = 1, \dots, g$) in (4.8) are generic points of the δ - \mathcal{F} -variety V . If $d > 0$, they also satisfy the equations*

$$u_{\sigma 0} + \sum_{\rho=1}^n u_{\sigma \rho} y_{\rho} = 0 \quad (\sigma = 1, \dots, d).$$

Proof. Suppose $\phi(y_1, \dots, y_n) \in \mathcal{F}\{\mathbb{Y}\}$ is any δ -polynomial vanishing on V . Then $\phi(\xi_1, \dots, \xi_n) = 0$. From Theorem 4.13, $\xi_{\rho} = \frac{\partial f}{\partial u_{0\rho}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}}$, we have

$$\phi\left(\frac{\partial f}{\partial u_{01}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}}, \dots, \frac{\partial f}{\partial u_{0n}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}}\right) = 0,$$

where $\frac{\partial f}{\partial u_{0\rho}^{(h)}}$ are obtained by substituting ζ_i to u_{i0} ($i = 0, 1, \dots, d$) in $\frac{\partial f}{\partial u_{0\rho}^{(h)}}$.

Hence, $\phi\left(\frac{\partial f}{\partial u_{01}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}}, \dots, \frac{\partial f}{\partial u_{0n}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}}\right)$ vanishes for $(u_{00}, \dots, u_{d0}) = (\zeta_0, \dots, \zeta_d)$.

Then there exists an $m \in \mathbb{N}$ such that $\left(\frac{\partial f}{\partial u_{00}^{(h)}}\right)^m \phi\left(\frac{\partial f}{\partial u_{01}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}}, \dots, \frac{\partial f}{\partial u_{0n}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}}\right) \in \text{sat}(f)$ in $\mathcal{F}\{\tilde{\mathbf{u}}, u_{00}\}$. By Lemma 4.25, we have $(f_{\tau 0})^m \phi(\xi_{\tau 1}, \dots, \xi_{\tau n}) = 0$. Thus $\phi(\xi_{\tau 1}, \dots, \xi_{\tau n}) = 0$, which means that $(\xi_{\tau 1}, \dots, \xi_{\tau n}) \in V$.

Conversely, for any $p \in \mathcal{F}\{\mathbb{Y}\}$ such that $p(\xi_{\tau 1}, \dots, \xi_{\tau n}) = 0$, there exists an $l \in \mathbb{N}$ such that $\tilde{p} = \left(\frac{\partial f}{\partial u_{00}^{(h)}}\right)^l p\left(\frac{\partial f}{\partial u_{01}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}}, \dots, \frac{\partial f}{\partial u_{0n}^{(h)}} / \frac{\partial f}{\partial u_{00}^{(h)}}\right)$ is in $\mathcal{F}\{\tilde{\mathbf{u}}, u_{00}\}$, which vanishes at $u_{00}^{(h)} = \gamma_{\tau}$. By Lemma 4.25, $\tilde{p} \in \text{sat}(f)$. Now treating \tilde{p} as a δ -polynomial in $\mathcal{F}\{\mathbf{u}\}\{u_{00}, \dots, u_{d0}\}$, we have $\tilde{p}(\xi_1, \dots, \xi_n) = 0$, and hence $p(\xi_1, \dots, \xi_n) = 0$. So $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ is a generic point of V .

Since $\frac{\partial f}{\partial u_{0\rho}^{(h)}} + \frac{\partial f}{\partial u_{00}^{(h)}}(-\xi_{\rho}) = 0$, we have $\sum_{\rho=1}^n u_{\sigma\rho} \frac{\partial f}{\partial u_{0\rho}^{(h)}} + \zeta_{\sigma} \frac{\partial f}{\partial u_{00}^{(h)}} = 0$. Thus, $\sum_{\rho=0}^n u_{\sigma\rho} \frac{\partial f}{\partial u_{0\rho}^{(h)}}$ vanishes at $(u_{00}, \dots, u_{d0}) = (\zeta_0, \dots, \zeta_d)$. In the case $\sigma \neq 0$, $\sum_{\rho=0}^n u_{\sigma\rho} \frac{\partial f}{\partial u_{0\rho}^{(h)}} = 0$. Consequently, $u_{\sigma 0} + \sum_{\rho=1}^n u_{\sigma\rho} \xi_{\tau\rho} = 0$ ($\sigma = 1, \dots, d$). □

Example 4.35. Continue from Example 4.30. We have $\xi_{11} = -u_{00}/u_{01}$ under the condition $u'_{00} = \gamma_1 = \frac{u_{00}}{u_{01}} u'_{01} - 2\sqrt{-1}\sqrt{u_{00}u_{01}}$. Then $\xi'_{11} = 2\sqrt{-\frac{u_{00}}{u_{01}}}$ and ξ_{11} is a zero of the original δ -ideal $\text{sat}(y_1'^2 - 4y_1)$.

Now, we will prove a result which gives the geometrical meaning of the leading δ -degree.

Suppose $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$ is the Chow form of V which is of dimension d , order h , and leading differential degree g . Recall that by saying a point $\eta = (\eta_1, \dots, \eta_n)$ lying on ${}^a\mathbb{P}_0^{(k)}$ we mean that $\eta^{[k]} = (\eta_1, \dots, \eta_n, \dots, \eta_1^{(k)}, \dots, \eta_n^{(k)})$ is an algebraic zero of ${}^a\mathbb{P}_0^{(k)}$. Theorem 4.34 and Corollary 4.29 show that $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ ($\tau = 1, \dots, g$) are intersection points of V and $\mathbb{P}_i = 0$ ($i = 1, \dots, d$) as well as ${}^a\mathbb{P}_0^{(k)} = 0$ ($k = 0, \dots, h-1$). In the next theorem, we will prove the converse of this result, that is, $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ ($\tau = 1, \dots, g$) are the only elements in V which are also on $\mathbb{P}_i = 0$ ($i = 1, \dots, d$) as well as on ${}^a\mathbb{P}_0^{(k)} = 0$ ($k = 0, \dots, h-1$). Intuitively, we use $\mathbb{P}_i = 0$ ($i = 1, \dots, d$) to decrease the dimension of V to zero and use ${}^a\mathbb{P}_0^{(k)} = 0$ ($k = 0, \dots, h-1$)

to determine the h arbitrary constants in the solutions of the zero-dimensional δ -variety.

Theorem 4.36. $(\xi_{\tau_1}, \dots, \xi_{\tau_n})$ ($\tau = 1, \dots, g$) defined in (4.8) are the only elements of V which also lie on \mathbb{P}_i ($i = 1, \dots, d$)³ as well as on ${}^a\mathbb{P}_0^{(j)}$ ($j = 0, \dots, h - 1$).

Proof. First, by Theorem 4.34 and Corollary 4.29, $(\xi_{\tau_1}, \dots, \xi_{\tau_n})$ ($\tau = 1, \dots, g$) are solutions of $\mathbb{I}(V)$ and $\mathbb{P}_1, \dots, \mathbb{P}_d$, which also lie on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$. It suffices to show that the number of solutions of $\mathbb{I}(V)$ and $\mathbb{P}_1, \dots, \mathbb{P}_d$, which also lie on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$, does not exceed g .

Let $\mathcal{J} = [\mathbb{I}(V), \mathbb{P}_1, \dots, \mathbb{P}_d] \subset \mathcal{F}^*\{\mathbb{Y}\}$, where $\mathcal{F}^* = \mathcal{F}\langle \bigcup_{i=1}^d \mathbf{u}_i \rangle$. By Theorem 3.14, \mathcal{J} is a prime δ -ideal of dimension zero and order h . Let $\mathcal{J}^{(h)} = \mathcal{J} \cap \mathcal{F}^*[\mathbb{Y}^{[h]}]$. Since \mathcal{J} is of dimension zero and order h , its δ -dimension polynomial is of the form $\omega(t) = h$, for $t \geq h$. So $\mathcal{J}^{(h)}$ is an algebraic prime ideal of dimension h .

Let $\mathcal{J}_0 = (\mathcal{J}^{(h)}, {}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)})$ be an algebraic ideal in the polynomial ring $\mathcal{F}_0[\mathbb{Y}^{[h]}, u_{00}, \dots, u_{00}^{(h-1)}]$, where $\mathcal{F}_0 = \mathcal{F}^*(\bigcup_{j=1}^n u_{0j}^{[h-1]})$ and $u_{0j}^{[h-1]}$ denote the set $\{u_{0j}, u'_{0j}, \dots, u_{0j}^{(h-1)}\}$. Similar to the proof of Lemma 3.5, we can show that \mathcal{J}_0 is an algebraic prime ideal of dimension h . If we can prove that $u_{00}^{[h-1]} = \{u_{00}, \dots, u_{00}^{(h-1)}\}$ is a parametric set of \mathcal{J}_0 , then it is clear that $\mathcal{J}_1 = (\mathcal{J}_0)$ is an algebraic prime ideal of dimension zero in $\mathcal{F}_0(u_{00}^{[h-1]})[\mathbb{Y}^{[h]}] = \mathcal{F}^*(\mathbf{u}_0^{[h-1]})[\mathbb{Y}^{[h]}]$. So we need to prove that $u_{00}^{[h-1]}$ is a parametric set of \mathcal{J}_0 . Suppose the contrary. Then there exists a nonzero δ -polynomial involving only $\{u_{00}, \dots, u_{00}^{(h-1)}\}$ as well as the other u , which belongs to \mathcal{J}_0 . Such a δ -polynomial also belongs to $[\mathcal{J}, \mathbb{P}_0] \in \mathcal{F}^*\langle u_{01}, \dots, u_{0n} \rangle \{y_1, \dots, y_n, u_{00}\}$. From the proof of Theorem 4.11, $\{F, S_F y_1 - \frac{\partial F}{\partial u_{01}^{(s)}}, \dots, S_F y_n - \frac{\partial F}{\partial u_{0n}^{(s)}}\}$ is a characteristic set of $[\mathcal{J}, \mathbb{P}_0]$ w.r.t. the elimination ranking $u_{00} \prec y_1 \prec \dots \prec y_n$. So this δ -polynomial can be reduced to zero by $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$. But $\text{ord}(F, u_{00}) = h$, a contradiction. So we have proved that \mathcal{J}_1 is an algebraic prime ideal of dimension zero.

Clearly, $\mathcal{J}_2 = (\mathcal{J}_1, {}^a\mathbb{P}_0^{(h)}) \in \mathcal{F}_0(u_{00}^{[h-1]}, \bigcup_{j=1}^n u_{0j}^{[h]})[\mathbb{Y}^{[h]}, u_{00}^{(h)}]$ is an algebraic prime ideal of dimension zero. Then, there exists an irreducible polynomial involving only $u_{ij}^{(k)}$ and $u_{00}^{(h)}$. Similar to above, it also belongs to $[\mathcal{J}, \mathbb{P}_0]$, thus it can be divisible by F . Since F is irreducible, it differs from F only by a factor in \mathcal{F} . Thus, $F = f(\mathbf{u}; u_{00}, u_{10}, \dots, u_{d0}) \in \mathcal{J}_2$.

Let (ξ_1, \dots, ξ_n) be a generic point of V and $\zeta_i = -\sum_{j=1}^n u_{ij} \xi_j$ ($i = 0, \dots, d$). By Lemma 4.3, the δ -ideal $[\mathbb{I}(V), \mathbb{P}_1, \dots, \mathbb{P}_d, \mathbb{P}_0]$ in $\mathcal{F}\langle \mathbf{u} \rangle \{y_1, \dots, y_n, u_{00}, \dots, u_{d0}\}$ has a generic point $(\xi_1, \dots, \xi_n, \zeta_0, \dots, \zeta_d)$. Since $f(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$, differentiating both sides of this identity w.r.t. $u_{0j}^{(k)}$, we have the following identities:

$$\frac{\partial f}{\partial u_{0j}^{(k)}} + \sum_{l=k}^h \frac{\partial f}{\partial u_{00}^{(l)}} \left(- \binom{l}{k} \xi_j^{(l-k)} \right) = 0, \quad (j = 1, \dots, n; k = 0, \dots, h),$$

where $\frac{\partial f}{\partial u_{0j}^{(k)}}$ are obtained by substituting ζ_i to u_{i0} in $\frac{\partial f}{\partial u_{0j}^{(k)}}$. Let $g_{jk} = \binom{h}{k} \frac{\partial f}{\partial u_{00}^{(h)}}$ $y_j^{(k)} + \sum_{l=1}^k \binom{h-l}{k-l} \frac{\partial f}{\partial u_{00}^{(h-l)}} y_j^{(k-l)} - \frac{\partial f}{\partial u_{0j}^{(h-k)}} (j = 1, \dots, n; k = 0, \dots, h)$. Then $g_{jk} \in [\mathbb{I}(V), \mathbb{P}_1,$

³If $d = 0$, \mathbb{P}_i ($i = 1, \dots, d$) is empty.

$\dots, \mathbb{P}_d, \mathbb{P}_0] \subset [\mathcal{J}, \mathbb{P}_0]$, for g_{jk} vanishes at $(\xi_1, \dots, \xi_n, \zeta_0, \dots, \zeta_d)$. Denote the algebraic ideal $[\mathcal{J}, \mathbb{P}_0] \cap \mathcal{F}^*(\bigcup_{j=1}^n u_{0j}^{[h]})[\mathbb{Y}^{[h]}, u_{00}, \dots, u_{00}^{(h)}]$ by $[\mathcal{J}, \mathbb{P}_0]^{(h)}$. It is clear that $g_{jk} \in [\mathcal{J}, \mathbb{P}_0]^{(h)}$. We will show that $[\mathcal{J}, \mathbb{P}_0]^{(h)} = (\mathcal{J}^{(h)}, {}^a\mathbb{P}_0, \dots, {}^a\mathbb{P}_0^{(h)})$, which implies that $g_{jk} \in \mathcal{J}_2$. Let $\eta = (\eta_1, \dots, \eta_n)$ be a generic point of \mathcal{J} . Then $(\eta_1, \dots, \eta_n, -\sum_{j=1}^n u_{0j}\eta_j)$ is a generic point of $[\mathcal{J}, \mathbb{P}_0]$. Thus,

$$(\eta_1, \dots, \eta_n, \dots, \eta_1^{(h)}, \dots, \eta_n^{(h)}, -\sum_{j=1}^n u_{0j}\eta_j, \dots, -\sum_{j=1}^n (u_{0j}\eta_j)^{(h)})$$

is a generic point of $[\mathcal{J}, \mathbb{P}_0]^{(h)}$. Of course, it is also a generic point of $(\mathcal{J}^{(h)}, {}^a\mathbb{P}_0, \dots, {}^a\mathbb{P}_0^{(h)})$. So the two ideals are identical. Thus, g_{jk} belongs to \mathcal{J}_2 . Note that the coefficient of $y_j^{(k)}$ in g_{jk} is $\binom{h}{k}S_f = \binom{h}{k}\frac{\partial f}{\partial u_{00}^{(h)}}$. So $\mathbb{V}(\mathcal{J}_2) \subseteq \mathbb{V}(f(\mathbb{U}_0, u_{00}^{(h)}), g_{jk} : j = 1, \dots, n; k = 0, \dots, h)$, and the latter algebraic variety consists of exactly g elements. Thus, $|\mathbb{V}(\mathcal{J}_1)| = |\mathbb{V}(\mathcal{J}_2)| \leq g$. Since every solution of \mathcal{J} which also lies on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$, when truncated up to order h , becomes a solution of \mathcal{J}_1 , it follows that the number of solutions of $\mathbb{I}(V)$ and $\mathbb{P}_1, \dots, \mathbb{P}_d$ which also lie on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$ does not exceed g . □

With Theorems 4.27, 4.32, 4.34, and 4.36, we proved the second and third statements of Theorem 1.2.

From the proof, we can see that a zero-dimensional algebraic ideal is obtained as shown by the following corollary.

Corollary 4.37. *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ of dimension d , order h , and leading δ -degree g . Use the same notation as Theorem 4.36. Then*

$$\tilde{\mathcal{I}} = ([\mathcal{I}, \mathbb{P}_1, \dots, \mathbb{P}_d] \cap \mathcal{F}^*[\mathbb{Y}^{[h]}, {}^a\mathbb{P}_0, \dots, {}^a\mathbb{P}_0^{(h-1)}]) \subset \mathcal{F}^*(\mathbf{u}_0^{[h-1]})[\mathbb{Y}^{[h]}]$$

is an algebraic prime ideal of dimension zero whose solutions are exactly $(\xi_{\tau 1}, \dots, \xi_{\tau n})^{[h]}$ ($\tau = 1, \dots, g$), where $\mathcal{F}^* = \mathcal{F}(\bigcup_{i=1}^d \mathbf{u}_i)$.

Example 4.38. Continue from Example 4.30. Note that $d = 0$, $h = 1$, and $g = 2$. Let V be the general component of $p = y_1^2 - 4y_1 = 0$. As in Theorem 4.36, we introduce the equation ${}^a\mathbb{P}_0 = u_{00} + u_{01}y_1$ which intersects V at two points: $\xi_{11} = -u_{00}/u_{01}$, with $\delta_1\xi_{11} = 2\sqrt{-u_{00}/u_{01}}$, and $\xi_{21} = -u_{00}/u_{01}$, with $\delta_2\xi_{21} = -2\sqrt{-u_{00}/u_{01}}$. As indicated by Corollary 4.37, $(\xi_{11}, \delta_1\xi_{11})$ and $(\xi_{21}, \delta_2\xi_{21})$ are the only solutions of the algebraic ideal $\tilde{\mathcal{I}} = (p, {}^a\mathbb{P}_0) \subset \mathbb{Q}(u_{00}, u_{01})[y_1, y'_1]$.

Due to Corollary 4.37, we can give a differential analog to Stickelberger’s Theorem in algebraic geometry.

Theorem 4.39 (Stickelberger’s Theorem [9, p.54], [33]). *Let $\mathcal{P} \subset \mathcal{F}[\mathbb{Y}]$ be a zero-dimensional ideal. Denote $A = \mathcal{F}[\mathbb{Y}]/\mathcal{P}$. Then A is a finite-dimensional vector space over \mathcal{F} . For any polynomial $f \in \mathcal{F}[\mathbb{Y}]$, let L_f be the \mathcal{F} -linear map:*

$$L_f : A \longrightarrow \frac{A}{\bar{g}}$$

where \bar{g} denotes the residue class of $g \in \mathcal{F}[\mathbb{Y}]$ in $A = \mathcal{F}[\mathbb{Y}]/\mathcal{P}$. Then the eigenvalues of L_f are $f(\alpha)$, with multiplicity m_α , where $\alpha \in \mathbb{V}(\mathcal{P})$ and m_α is the multiplicity of $\alpha \in \mathbb{V}(\mathcal{P})$. Thus, the determinant of L_f is $\prod_{\alpha \in \mathbb{V}(\mathcal{P})} f(\alpha)^{m_\alpha}$.

Theorem 4.40 (Differential Stickelberger’s Theorem). *Let \mathcal{I} be a prime δ -ideal in $\mathcal{F}\{\mathbb{Y}\}$ of dimension zero, order h , and leading δ -degree g . Let $\mathbb{P}_0 = u_{00} + u_{01}y_1 + \dots + u_{0n}y_n = 0$ be a generic δ -hyperplane and ${}^a\mathbb{P}_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$ be defined as in (4.10). For any δ -polynomial $p \in \mathcal{F}\{\mathbb{Y}\}$, let $s = \max\{h, \text{ord}(p)\}$ and L_p be the $\mathcal{F}(\mathbf{u}_0^{[h-1]})$ -linear map:*

$$L_p : \mathcal{F}(\mathbf{u}_0^{[h-1]})[\mathbb{Y}^{[s]}/\tilde{\mathcal{I}}]_{\bar{g}} \longrightarrow \mathcal{F}(\mathbf{u}_0^{[h-1]})[\mathbb{Y}^{[s]}/\tilde{\mathcal{I}}]_{\overline{gp}}$$

where $\tilde{\mathcal{I}} = (\mathcal{I} \cap \mathcal{F}[\mathbb{Y}^{[s]}, {}^a\mathbb{P}_0, \dots, {}^a\mathbb{P}_0^{(h-1)}]) \subset \mathcal{F}(\mathbf{u}_0^{[h-1]})[\mathbb{Y}^{[s]}$ and \bar{g} denotes the residue class of $g \in \mathcal{F}(\mathbf{u}_0^{[h-1]})[\mathbb{Y}^{[s]}$ in $\mathcal{F}(\mathbf{u}_0^{[h-1]})[\mathbb{Y}^{[s]}/\tilde{\mathcal{I}}$. Then the eigenvalues of L_p are $p(\xi_\tau)$ and the determinant of L_p is $\prod_{\tau=1}^g p(\xi_\tau)$.

Proof. By Theorem 4.39, it suffices to show that $\tilde{\mathcal{I}}$ is a prime δ -ideal in $\mathcal{F}(\mathbf{u}_0^{[h-1]})[\mathbb{Y}^{[s]}$ of dimension zero, and $\mathbb{V}(\tilde{\mathcal{I}}) = \{\xi_\tau^{[s]} = (\xi_{\tau 1}, \dots, \xi_{\tau n}, \dots, \delta_\tau^s \xi_{\tau 1}, \dots, \delta_\tau^s \xi_{\tau n}) : \tau = 1, \dots, g\}$. If $s = h$, then this is a direct consequence of Corollary 4.37. If $s > h$, $\xi_\tau^{[s]}$ clearly vanishes ${}^a\mathbb{P}_0^{(i)}$ ($i = 0, \dots, h - 1$). By Theorem 4.36, $\xi_\tau^{[s]}$ are also zeros of $\mathcal{I} \cap \mathcal{F}[\mathbb{Y}^{[s]}$ and are the only zeros of $\tilde{\mathcal{I}}$. □

4.6. Relations between the differential Chow form and the variety. In the algebraic case, we can obtain the defining equations of a variety from its Chow form. But in the differential case, this is not valid. Now we proceed as follows to obtain a weaker result. Recall that a δ -variety is unmixed if all of its components have the same dimension.

Lemma 4.41. *Let V be an irreducible δ - \mathcal{F} -variety of dimension $d > 0$ and $(0, \dots, 0) \notin V$. Then, the intersection of V with a generic δ -hyperplane passing through $(0, \dots, 0)$ is either empty or unmixed of dimension $d - 1$. Moreover, in the case $d > 1$, it is exactly unmixed of dimension $d - 1$.*

Proof. Let $\mathcal{I} = \mathbb{I}(V)$ be the prime δ - \mathcal{F} -ideal corresponding to V . A generic δ -hyperplane passing through $(0, \dots, 0)$ is $u_1y_1 + u_2y_2 + \dots + u_ny_n$, where the $u_i \in \mathcal{E}$ are δ - \mathcal{F} -indeterminates. Since $(0, 0, \dots, 0) \notin V$, we have

$$\begin{aligned} & V \cap \mathbb{V}(u_1y_1 + u_2y_2 + \dots + u_ny_n) \\ &= \mathbb{V}(\mathcal{I}, u_1y_1 + u_2y_2 + \dots + u_ny_n) \\ &= \bigcup_{i=1}^n \mathbb{V}([\mathcal{I}, u_1y_1 + u_2y_2 + \dots + u_ny_n]/y_i) \\ &= \bigcup_{i=1}^n \mathbb{V}([\mathcal{I}, u_1y_1 + u_2y_2 + \dots + u_ny_n] : y_i^\infty). \end{aligned}$$

Suppose a generic point of V is (ξ_1, \dots, ξ_n) . Since $(0, 0, \dots, 0) \notin V$, there exists at least one $i \in \{1, \dots, n\}$ such that $\xi_i \neq 0$. Of course, $\xi_i = 0$ means $\mathbb{V}([\mathcal{I}, u_1y_1 + u_2y_2 + \dots + u_ny_n] : y_i^\infty) = \emptyset$. So we need only consider the case when $\xi_i \neq 0$. Without loss of generality, we suppose $\xi_1 \neq 0$.

Let

$$\mathcal{Q} = [\mathcal{I}, u_1y_1 + u_2y_2 + \dots + u_ny_n] : y_1^\infty \subseteq \mathcal{F}\langle u_1, \dots, u_n \rangle \{y_1, \dots, y_n\}$$

and

$$\mathcal{Q}_0 = [\mathcal{I}, u_1y_1 + u_2y_2 + \dots + u_ny_n] : y_1^\infty \subseteq \mathcal{F}\langle u_2, \dots, u_n \rangle \{y_1, \dots, y_n, u_1\}.$$

It is easy to verify that $(\xi_1, \dots, \xi_n, -\frac{u_2\xi_2+\dots+u_n\xi_n}{\xi_1})$ is a generic point of \mathcal{Q}_0 and $\dim(\mathcal{Q}_0) = d$ by following the proof of Lemma 3.5. Now we discuss it in three cases.

Case 1. $\mathcal{I} \cap \mathcal{F}\{y_1\} \neq \{0\}$, that is, ξ_1 is δ -algebraic over \mathcal{F} . We have

$$\dim V = d = \text{d.tr.deg } \mathcal{F}\langle \xi_1, \dots, \xi_n \rangle / \mathcal{F} = \text{d.tr.deg } \mathcal{F}\langle \xi_1 \rangle \langle \xi_2, \dots, \xi_n \rangle / \mathcal{F}\langle \xi_1 \rangle.$$

Suppose ξ_2, \dots, ξ_{d+1} are δ -independent over $\mathcal{F}\langle \xi_1 \rangle$.

First, $\mathcal{Q}_0 \cap \mathcal{F}\langle u_2, \dots, u_n \rangle \{u_1\} = \{0\}$. For if not, we have a nonzero δ -polynomial $h(u_2, \dots, u_n, u_1) \in \mathcal{F}\{u_2, \dots, u_n, u_1\}$ such that $h(u_2, \dots, u_n, -\frac{u_2\xi_2+\dots+u_n\xi_n}{\xi_1}) = 0$. For a fixed i between 2 and n , if we specialize u_i to -1 and u_j ($j \neq i$) to 0, then by Theorem 2.16, ξ_i/ξ_1 is δ -algebraic over \mathcal{F} . So each ξ_i ($i = 1, \dots, n$) is δ -algebraic over \mathcal{F} , which contradicts the fact that $d > 0$. It follows that \mathcal{Q} is not the unit ideal and $\dim(\mathcal{Q}) \geq 0$.

Second, since y_2, \dots, y_{d+1} is a parametric set of \mathcal{I} , it is also a parametric set for \mathcal{Q}_0 . So y_2, \dots, y_{d+1}, u_1 are δ -dependent modulo \mathcal{Q}_0 . Since $\mathcal{Q}_0 \cap \mathcal{F}\langle u_2, \dots, u_n \rangle \{u_1\} = \{0\}$, we know that y_2, \dots, y_{d+1} are δ -dependent modulo \mathcal{Q} . Using the fact that each remaining y_i and y_2, \dots, y_{d+1} are δ -dependent modulo \mathcal{Q} , we obtain $\dim(\mathcal{Q}) \leq d-1$. If $d = 1$, then $\dim(\mathcal{Q}) = 0 = d-1$ follows. Now for $d > 1$, we claim that $\dim(\mathcal{Q}) = d-1$ by proving that y_2, \dots, y_d are δ -independent modulo \mathcal{Q} . For if not, there exists $0 \neq h(y_2, \dots, y_d, u_1) \in \mathcal{Q}_0$ such that $h(\xi_2, \dots, \xi_d, -\frac{u_2\xi_2+\dots+u_n\xi_n}{\xi_1}) = 0$. By Theorem 2.16, we can specialize u_{d+1} to -1 , the other u_i to zero, and conclude that $\xi_2, \dots, \xi_d, \frac{\xi_{d+1}}{\xi_1}$ are δ -dependent over \mathcal{F} . Since ξ_2, \dots, ξ_d are δ -independent over \mathcal{F} , ξ_{d+1} is δ -algebraic over $\mathcal{F}\langle \xi_1, \dots, \xi_d \rangle$, which is a contradiction. Thus $\dim \mathcal{Q} = d-1$.

Case 2. $d > 1$ and ξ_1 is δ -transcendental over \mathcal{F} . In this case, we suppose a δ -transcendence basis is ξ_1, \dots, ξ_d .

First, $\mathcal{Q}_0 \cap \mathcal{F}\langle u_2, \dots, u_n \rangle \{u_1\} = \{0\}$. For if not, as in the preceding case, we conclude that ξ_i/ξ_1 is δ -algebraic over \mathcal{F} , that is, ξ_i, ξ_1 are δ -algebraic over \mathcal{F} , which contradicts the fact that $d > 1$. So \mathcal{Q} is a nontrivial prime δ -ideal.

Second, $\dim(\mathcal{Q}) = d-1$, for on the one hand from the fact that $y_1, y_2, \dots, y_d, u_1$ are δ -dependent modulo \mathcal{Q}_0 , we have $\dim(\mathcal{Q}) \leq d-1$, and on the other hand, from the fact that y_2, \dots, y_d, u_1 are δ -independent modulo \mathcal{Q}_0 , there comes $\dim(\mathcal{Q}) \geq d-1$.

Case 3. $d = 1$ and ξ_1 is δ -transcendental over \mathcal{F} . If $\mathcal{Q}_0 \cap \mathcal{F}\langle u_2, \dots, u_n \rangle \{u_1\} \neq \{0\}$, the intersection is empty. If $\mathcal{Q} \neq [1]$, similar to Case 2, we can easily prove that the intersection is of dimension zero.

So for each $i \in \{1, \dots, n\}$ such that $\xi_i \neq 0$, we can show that $\mathbb{V}([\mathcal{I}, u_1y_1 + u_2y_2 + \dots + u_ny_n] : y_i^\infty)$ is either empty or of dimension $d-1$, similar to the above steps for the case $i = 1$. If $d > 1$, it is exactly of dimension $d-1$. Thus the theorem is proved. □

By saying independent generic δ -hyperplanes, we mean that the coefficients of these δ -hyperplanes are δ -indeterminates in \mathcal{E} . The following result gives an equivalent condition for a point to be in a δ -variety.

Theorem 4.42. *Let V be a δ - \mathcal{F} -variety of dimension d . Then $\bar{x} \in V$ if and only if $d+1$ independent generic δ -hyperplanes $\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_d$ passing through \bar{x} meet V .*

Proof. The necessity of the condition is obviously true. We now consider the sufficiency. We adjoin the coordinates of \bar{x} to \mathcal{F} , and denote $\overline{\mathcal{F}}$ to be the δ -field thus obtained. Regarded as a δ -variety over $\overline{\mathcal{F}}$, V is the sum of a finite number of irreducible δ -varieties \overline{V}_i , which are of dimension d [32, p.51]. Suppose $\bar{x} \notin V$, and therefore does not lie in any component of V . We now prove that any $d + 1$ independent generic δ -hyperplanes passing through \bar{x} do not meet \overline{V}_i . Without loss of generality, suppose $\bar{x} = (0, 0, \dots, 0)$. Then a generic δ -hyperplane passing through \bar{x} is $s_1y_1 + \dots + s_ny_n$, where $s_i \in \mathcal{E}$ are δ - \mathcal{F} -indeterminates. We proceed by induction on d .

If $d = 0$, then for $(a_1, \dots, a_n) \in V$, each a_i is δ -algebraic over \mathcal{F} . If $V \cap \mathbb{V}(s_1y_1 + \dots + s_ny_n) \neq \emptyset$, then there exists some $(a_1, \dots, a_n) \in V$ such that $s_1a_1 + \dots + s_na_n = 0$, since the s_i are δ -independent over \mathcal{F} . Thus $(a_1, \dots, a_n) = (0, \dots, 0)$, a contradiction to the fact that $\bar{x} \notin V$. Thus the theorem is proved when $d = 0$.

We therefore assume the truth of the theorem for δ -varieties of dimension less than d , and consider a δ -variety V of dimension d . Let $\mathbb{P}_0, \dots, \mathbb{P}_d$ be $d + 1$ independent generic δ -hyperplanes passing through \bar{x} . The equation \mathbb{P}_d can be written as $s_1y_1 + \dots + s_ny_n = 0$ with s_i δ - \mathcal{F} -indeterminates. From Lemma 4.41, $\mathbb{P}_d = 0$ meets V in a δ -variety \mathbb{W} of dimension less than d . By the hypothesis of the induction, $\mathbb{P}_0, \dots, \mathbb{P}_{d-1}$ do not meet \mathbb{W} ; it follows that V does not meet $\mathbb{P}_0, \dots, \mathbb{P}_d$. Therefore the theorem is proved. \square

The following result proves the fourth statement of Theorem 1.2.

Theorem 4.43. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ be the Chow form of V and $S_F = \frac{\partial F}{\partial u_{00}^{(h)}}$. Suppose that \mathbf{u}_i ($i = 0, \dots, d$) are δ -specialized over \mathcal{F} to sets \mathbf{v}_i of specific elements in \mathcal{E} and $\overline{\mathbb{P}}_i$ ($i = 0, \dots, d$) are obtained by substituting \mathbf{u}_i by \mathbf{v}_i in \mathbb{P}_i . If $\overline{\mathbb{P}}_i = 0$ ($i = 0, \dots, d$) meet V , then $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$. Furthermore, if $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$ and $S_F(\mathbf{v}_0, \dots, \mathbf{v}_d) \neq 0$, then the $d + 1$ δ -hyperplanes $\overline{\mathbb{P}}_i = 0$ ($i = 0, \dots, d$) meet V .*

Proof. Let $\mathcal{I} = \mathbb{I}(V) \subseteq \mathcal{F}\{\mathbb{Y}\}$, $\mathbb{I}_{\zeta, \xi} = [\mathcal{I}, \mathbb{P}_0, \dots, \mathbb{P}_d] \subseteq \mathcal{F}\langle \mathbf{u} \rangle \{y_1, \dots, y_n, u_{00}, \dots, u_{d0}\}$, and $\mathcal{I}_1 = [\mathbb{I}_{\zeta, \xi}] \subseteq \mathcal{F}\langle \mathbf{u}_0, \dots, \mathbf{u}_d \rangle \{y_1, \dots, y_n\}$. By Lemma 4.10, $\{F, \frac{\partial F}{\partial u_{00}^{(h)}}y_1 - \frac{\partial F}{\partial u_{01}^{(h)}}, \dots, \frac{\partial F}{\partial u_{00}^{(h)}}y_n - \frac{\partial F}{\partial u_{0n}^{(h)}}\}$ is a characteristic set of $\mathbb{I}_{\zeta, \xi}$ w.r.t. the elimination ranking $u_{d0} \prec \dots \prec u_{00} \prec y_1 \prec \dots \prec y_n$. Since F is irreducible, $\mathbb{I}_{\zeta, \xi} = [F, S_Fy_1 - \frac{\partial F}{\partial u_{01}^{(h)}}, \dots, S_Fy_n - \frac{\partial F}{\partial u_{0n}^{(h)}}] : S_F^\infty$ with $S_F = \frac{\partial F}{\partial u_{00}^{(h)}}$.

When \mathbf{u}_i are δ -specialized to \mathbf{v}_i , \mathcal{I}_1 becomes a δ -ideal in $\mathcal{F}\langle \mathbf{v}_0, \dots, \mathbf{v}_d \rangle \{\mathbb{Y}\}$. If $\overline{\mathbb{P}}_0, \dots, \overline{\mathbb{P}}_d$ meet V , then $\overline{\mathcal{I}}_1 = [\mathcal{I}, \overline{\mathbb{P}}_0, \dots, \overline{\mathbb{P}}_d] \neq [1]$, which implies $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$ since $F \in \mathbb{I}_{\zeta, \xi}$.

If $S_F(\mathbf{v}_0, \dots, \mathbf{v}_d) \neq 0$ and $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$, then let $\overline{y}_i = (\frac{\partial F}{\partial u_{0i}^{(h)}}(\mathbf{v}_0, \dots, \mathbf{v}_d)) / (S_F(\mathbf{v}_0, \dots, \mathbf{v}_d))$ ($i = 1, \dots, n$). We claim that $(\overline{y}_1, \dots, \overline{y}_n)$ lies in V and the $d + 1$ δ -hyperplanes $\overline{\mathbb{P}}_0, \dots, \overline{\mathbb{P}}_d$, which implies that $\overline{\mathbb{P}}_0, \dots, \overline{\mathbb{P}}_d$ meet V .

First, let p be any δ -polynomial in \mathcal{I} . Then $p \in \mathbb{I}_{\zeta, \xi}$, so there exists an integer m such that $S_F^m p \in [F, S_Fy_1 - \frac{\partial F}{\partial u_{01}^{(h)}}, \dots, S_Fy_n - \frac{\partial F}{\partial u_{0n}^{(h)}}]$. If we specialize $u_{ij} \rightarrow v_{ij}$, $u_{i0} \rightarrow v_{i0}$ and let $y_i = \overline{y}_i$, then we have $S_F^m(\mathbf{v}_0, \dots, \mathbf{v}_d)p(\overline{y}_1, \dots, \overline{y}_n) = 0$, so $p(\overline{y}_1, \dots, \overline{y}_n) = 0$. That is, $(\overline{y}_1, \dots, \overline{y}_n) \in V$. Second, since $\mathbb{P}_i \in \mathbb{I}_{\zeta, \xi}$, similar to the above, it follows that $(\overline{y}_1, \dots, \overline{y}_n)$ lies in $\overline{\mathbb{P}}_i$. So $\overline{\mathbb{P}}_0, \dots, \overline{\mathbb{P}}_d$ meet V . \square

Remark 4.44. Let X be the set of all $(n - d - 1)$ -dimensional linear spaces in \mathcal{F}^n that meet an irreducible δ - \mathcal{F} -variety V of dimension d . From Theorem 4.43, $X \subset \mathbb{V}(F)$ and $X \setminus \mathbb{V}(S_F) = \mathbb{V}(F) \setminus \mathbb{V}(S_F)$. That is, a “major” part of X is known to be $\mathbb{V}(F) \setminus \mathbb{V}(S_F)$. An interesting problem is to see whether X is a δ -variety for a projective δ -variety V . In [24], Kolchin showed that this problem has a positive answer in a special case; that is, V is a projective algebraic variety which is treated as a zero-dimensional differential projective variety in a certain way.

Similar to the algebraic case [16, p.22], we can show that a generic δ -hyperplane passing through a given point $x = (x_1, x_2, \dots, x_n)$ is of the form $a_0 + a_1y_1 + \dots + a_ny_n = 0$ with $a_i = \sum_{j=0}^n s_{ij}x_j (i = 0, 1, \dots, n)$, where $x_0 = 1$ and $S = (s_{ij})$ is an $(n + 1) \times (n + 1)$ skew-symmetric matrix with $s_{ij} (i < j)$ independent δ - \mathcal{F} -indeterminates in \mathcal{E} . That is,

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = S \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For convenience, we denote such a δ -hyperplane by Sx and say a generic δ -hyperplane passing through a point x is of the form Sx .

Now we write $\mathbf{u}_i = (u_{i0}, u_{i1}, \dots, u_{in})^T = S^iY$, where $Y = (1, y_1, \dots, y_n)^T$ and the S^i are skew-symmetric matrices with $s_{jk}^i (j < k)$ independent δ - \mathcal{F} -indeterminates in \mathcal{E} . Substituting the \mathbf{u}_i in $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ by these equations, we obtain a δ -polynomial involving $s_{jk}^i (j < k)$ and the y_l . Regarding this δ -polynomial as a δ -polynomial in $s_{jk}^i (j < k)$, then we have $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = F(S^0Y, S^1Y, \dots, S^dY) = \sum g_\phi(y_1, \dots, y_n)\phi(s_{jk}^i)$, where $\phi(s_{jk}^i)$ are different δ -monomials. In this way, we get a finite number of δ -polynomials $g_\phi(y_1, \dots, y_n)$ over \mathcal{F} , which is denoted by \mathcal{P} . Similarly, in this way, we will get another set \mathcal{D} of δ -polynomials from $S_F(\mathbf{u}_0, \dots, \mathbf{u}_d)$.

Theorem 4.45. *Let V be an irreducible δ - \mathcal{F} -variety with dimension d and $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ its Chow form. Then $V \setminus \mathbb{V}(\mathcal{D}) = \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D}) \neq \emptyset$, where \mathcal{P}, \mathcal{D} are the δ -polynomial sets obtained from $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ and $S_F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ respectively as above.*

Proof. On the one hand, for any $\bar{x} \in V$, from Theorem 4.42, any $d + 1$ generic δ -hyperplanes passing through \bar{x} meet V . So $S^0\bar{x}, S^1\bar{x}, \dots, S^d\bar{x}$ meet V . By the proof of Theorem 4.43, $F(S^0\bar{x}, S^1\bar{x}, \dots, S^d\bar{x}) = 0$. Since $s_{jk}^i (j < k)$ are δ -ndeterminates, $\bar{x} \in \mathbb{V}(\mathcal{P})$. So $V \setminus \mathbb{V}(\mathcal{D}) \subseteq \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D})$.

On the other hand, for any $\bar{x} \in \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D})$, since any $d + 1$ generic δ -hyperplanes passing through \bar{x} are of the form $S^0\bar{x}, S^1\bar{x}, \dots, S^d\bar{x}$ with the S^i δ -indeterminate skew-symmetric matrices, we have $F(S^0\bar{x}, S^1\bar{x}, \dots, S^d\bar{x}) = 0$ and $S_F(S^0\bar{x}, S^1\bar{x}, \dots, S^d\bar{x}) \neq 0$. From Theorem 4.43, $S^0\bar{x}, S^1\bar{x}, \dots, S^d\bar{x}$ meet V . Thus from Theorem 4.42, $\bar{x} \in V$. Thus $V \setminus \mathbb{V}(\mathcal{D}) = \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D})$.

Now, we show that $V \setminus \mathbb{V}(\mathcal{D}) \neq \emptyset$. Suppose the contrary, i.e. $V \subset \mathbb{V}(\mathcal{D})$, in particular, its generic point $(\xi_1, \dots, \xi_n) \in \mathbb{V}(\mathcal{D})$. Thus, $S_F(S^0\xi, S^1\xi, \dots, S^d\xi) = 0$, where $\xi = (1, \xi_1, \dots, \xi_n)$. Recall that $s_{jk}^i (j < k; i = 0, 1, \dots, d)$ are independent δ -indeterminates over $\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle$. Now we consider a δ -endomorphism ϕ of $\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle\{s_{jk}^i (j < k; i = 0, 1, \dots, d)\}$ over $\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle$ satisfying $\phi(s_{0k}^i) =$

$-s_{0k}^i$ and $\phi(s_{jk}^i) = 0$ ($j < k; j = 1, \dots, n$). It is clear that $\phi(S_F(S^0\xi, \dots, S^d\xi)) = S_F(s_{0k}^i; -\sum_{k=1}^n s_{0k}^0 \xi_k, \dots, -\sum_{k=1}^n s_{0k}^d \xi_k) = 0$. Denoting s_{0k}^i by u_{ik} , we have $S_F(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$, thus $S_F \in \text{sat}(F)$, which is a contradiction. So $V \setminus \mathbb{V}(\mathcal{D}) \neq \emptyset$. \square

Since V is an irreducible δ -variety, $V \cap \mathbb{V}(\mathcal{D})$ is a subset of V with lower dimension than that of V or with the same dimension but of lower order. Thus, $V \setminus \mathbb{V}(\mathcal{D})$ is an open set of V in the Kolchin topology.

Example 4.46. Continue from Example 4.5. In this example, $F(\mathbf{u}_0) = u_1^2(u'_0)^2 - 2u_1u'_1u_0u'_0 + (u'_1)^2u_0^2 + 4u_1^3u_0$ and $S_F(\mathbf{u}_0) = 2u_1^2u'_0 - 2u_1u'_1u_0$. Following the steps as above, we obtain $\mathcal{P} = \{(y'_1)^2 - 4y_1\}$ and $\mathcal{D} = \{y'_1\}$. That is, we obtain the defining equation $(y'_1)^2 - 4y_1 = 0$ for the δ -variety under the condition $y'_1 \neq 0$.

5. DIFFERENTIAL CHOW VARIETY

In Theorem 1.2, we have listed four properties for the differential Chow form. In this section, we are going to prove that these properties are also the sufficient conditions for a δ -polynomial $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$ to be the Chow form for a δ -variety. Based on these sufficient conditions, we can define the δ -Chow quasi-variety for certain classes of δ -varieties in the sense that a point in the δ -Chow quasi-variety represents a δ -variety in the class. In other words, we give a parametrization of all δ -varieties in the class. Obviously, this is an extension of the concept of Chow variety in the algebraic case [13, 16].

5.1. Sufficient conditions for a polynomial to be a differential Chow form.

The following result gives sufficient conditions for a δ -polynomial to be the Chow form of an irreducible δ -variety. From Theorem 1.2, they are also necessary conditions.

Theorem 5.1. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ be an irreducible δ -polynomial in $\mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d\}$, where $\mathbf{u}_i = (u_{i0}, u_{i1}, \dots, u_{in})$ ($i = 0, \dots, d$). If F satisfies the following conditions, then it is the Chow form for an irreducible δ -variety of dimension d and order h .*

1. $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ is δ -homogenous of the same degree in each set of δ -variables \mathbf{u}_i and $\text{ord}(F, u_{ij}) = h$ for all u_{ij} occurring in F .
2. $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ can be factored uniquely into the following form:

$$\begin{aligned} F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) &= A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) \prod_{\tau=1}^g (u_{00}^{(h)} + \sum_{\rho=1}^n u_{0\rho}^{(h)} \xi_{\tau\rho} + t_\tau) \\ &= A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) \prod_{\tau=1}^g (u_{00} + \sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})^{(h)}, \end{aligned}$$

where $g = \text{deg}(F, u_{00}^{(h)})$ and $\xi_{\tau\rho}$ are in a differential extension field \mathcal{F}_τ of \mathcal{F} . The first “=” is obtained by factoring $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ as an algebraic polynomial in the variables $u_{00}^{(h)}, u_{01}^{(h)}, \dots, u_{0n}^{(h)}$, while the second one is a differential expression by defining the derivatives of $\xi_{\tau\rho}$ to be

$$\xi_{\tau\rho}^{(m)} = (\delta \xi_{\tau\rho}^{(m-1)})|_{u_{00}^{(h)} = -\sum_{\rho=1}^n u_{0\rho}^{(h)} \xi_{\tau\rho} - t_\tau} \quad (m \geq 1)$$

recursively.

3. $\Xi_\tau = (\xi_{\tau 1}, \dots, \xi_{\tau n})$ ($\tau = 1, \dots, g$) are on the δ -hyperplanes $\mathbb{P}_\sigma = 0$ ($\sigma = 1, \dots, d$) as well as on the algebraic hyperplanes ${}^a\mathbb{P}_0^{(k)} = 0$ ($k = 0, \dots, h - 1$).

4. For each τ , if $v_{i0} + v_{i1}\xi_{\tau 1} + \dots + v_{in}\xi_{\tau n} = 0$ ($i = 0, \dots, d$), then $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$, where $\mathbf{v}_i = (v_{i0}, v_{i1}, \dots, v_{in})$ and $v_{ij} \in \mathcal{E}$. Equivalently, if S^0, \dots, S^d are $(n + 1) \times (n + 1)$ skew-symmetric matrices, each having independent δ -indeterminates above its principle diagonal, then $F(S^0\xi_\tau, \dots, S^d\xi_\tau) = 0$, where $\xi_\tau = (1, \xi_{\tau 1}, \dots, \xi_{\tau n})$.

Before proving the theorem, we need several lemmas.

Lemma 5.2 ([15, p.11, Theorem 1]). *Let \mathcal{R} and \mathcal{S}^* be two rings and \mathcal{R} isomorphic to a subring \mathcal{S} of \mathcal{S}^* . Then there exists an extension ring \mathcal{R}^* of \mathcal{R} such that this isomorphism can be extended to an isomorphism between \mathcal{R}^* and \mathcal{S}^* .*

Lemma 5.3. *Let V be an irreducible δ - \mathcal{F} -variety of dimension $d > 0$ and $\mathbb{P} = u_0 + u_1y_1 + \dots + u_ny_n$ be a generic δ -hyperplane where $u_i \in \mathcal{E}$. Then every generic point of $\mathbb{V}(\mathbb{I}(V), \mathbb{P})$ over $\mathcal{F}_1 = \mathcal{F}\langle u_0, \dots, u_n \rangle$ is a generic point of V over \mathcal{F} .*

Proof. By Theorem 3.7, $[\mathbb{I}(V), \mathbb{P}]$ is a prime δ -ideal of dimension $d - 1$ in $\mathcal{F}_1\{\mathbb{Y}\}$. Let η be a generic point of $\mathbb{V}(\mathbb{I}(V), \mathbb{P})$. Then for any δ -polynomial p in $\mathbb{I}(V)$, we have $p(\eta) = 0$. On the other hand, for any δ -polynomial $p \in \mathcal{F}\{y_1, \dots, y_n\}$ such that $p(\eta) = 0$, we have $p \in [\mathbb{I}(V), \mathbb{P}]$. Then $p \equiv \sum_i h_i \mathbb{P}^{(i)} \pmod{\mathbb{I}(V)}$. Substituting u_0 by $-u_1y_1 - \dots - u_ny_n$ in the above equality, we have $p \equiv 0 \pmod{\mathbb{I}(V)}$. Hence η is a generic point of V . □

In the next result, we will show that if condition 4 from Theorem 5.1 holds, then the following stronger version is also valid.

Lemma 5.4. *Let F satisfy condition 4 of Theorem 5.1. Consider F as an algebraic polynomial $f(u_{\sigma j}^{(k)}, u_{0j}^{(l)}, u_{00}^{(h)}, \dots, u_{0n}^{(h)})$ in $u_{ij}^{(k)}$, and $(\xi_{\tau 1}, \dots, \xi_{\tau n}, \dots, \xi_{\tau 1}^{(h)}, \dots, \xi_{\tau n}^{(h)})$ is regarded as an algebraic point. If $w_{i0k} + \sum_{j=1}^n \sum_{m=0}^k \binom{k}{m} w_{ijm} \delta_\tau^{(k-m)} \xi_{\tau j} = 0$ ($i = 0, \dots, d; k = 0, \dots, h$), then $f(w_{\sigma jk}, w_{0jl}, w_{00h}, \dots, w_{0nh}) = 0$, where the w_{ijk} are considered as elements in the underlying ordinary field of \mathcal{E} .*

Proof. Regard $\mathcal{Q}_\tau = [v_{i0} + \sum_{j=1}^n v_{ij}\xi_{\tau j} : i = 0, \dots, d]$ as a δ -ideal in $\mathcal{F}\langle \xi_{\tau j} : j = 1, \dots, n \rangle\{v_{i0}, \dots, v_{in} : i = 0, \dots, d\}$, where v_{ij} are δ - $\mathcal{F}\langle \xi_{\tau j} \rangle$ -indeterminates. From condition 4 of Theorem 5.1, $F(\mathbf{v}_0, \dots, \mathbf{v}_d)|_{\mathbb{V}(\mathcal{Q}_\tau)} \equiv 0$. It is clear that \mathcal{Q}_τ is a prime δ -ideal and $\{v_{i0} + \sum_{j=1}^n v_{ij}\xi_{\tau j} : i = 0, \dots, d\}$ is its characteristic set with v_{i0} as leaders. By the differential Nullstellensatz, $F(\mathbf{v}_0, \dots, \mathbf{v}_d) \in \mathcal{Q}_\tau$. From condition 1 of Theorem 5.1, $\text{ord}(F, v_{i0}) = h$. Then $F(\mathbf{v}_0, \dots, \mathbf{v}_d) \in (v_{i0} + \sum_{j=1}^n v_{ij}\xi_{\tau j}, \dots, v_{i0}^{(h)} + \sum_{j=1}^n v_{ij}^{(h)}\xi_{\tau j} + \sum_{j=1}^n \sum_{m=0}^{h-1} \binom{h}{m} v_{ij}^{(m)} \xi_{\tau j}^{(h-m)} : i = 0, \dots, d)$. Regarding the above relation as a pure algebraic relation, we can substitute $v_{ij}^{(k)}$ by algebraic indeterminates w_{ijk} and regard F as an algebraic polynomial. Then $f(w_{\sigma jk}, w_{0jl}, w_{00h}, \dots, w_{0nh}) \in (w_{i00} + \sum_{j=1}^n w_{ij0}\xi_{\tau j}, \dots, w_{i0h} + \sum_{j=1}^n w_{ijh}\xi_{\tau j} + \sum_{j=1}^n \sum_{m=0}^{h-1} \binom{h}{m} w_{ijm}\xi_{\tau j}^{(h-m)} : i = 0, \dots, d)$, which shows that the lemma is valid. □

Proof of Theorem 5.1. Let V_τ ($\tau = 1, \dots, g$) be the irreducible δ - \mathcal{F} -variety with $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ as its generic point over \mathcal{F} . We will later show that all the δ -varieties V_τ are the same.

First, we claim that the generic points of V_τ which lie on $\mathbb{P}_1, \dots, \mathbb{P}_d$ as well as on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$ are included in $\{(\xi_{\tau 1}, \dots, \xi_{\tau n}) : \tau = 1, \dots, g\}$. Without

loss of generality, we consider V_1 . To prove that the claim holds for V_1 , we need to use the following assertion:

(\star) If $(\eta_{10}, \dots, \eta_{n0}, \dots, \eta_{1h}, \dots, \eta_{nh})$ is a generic point of the algebraic ideal $\mathbb{I}(V_1)^{(h)} = \mathbb{I}(V_1) \cap \mathcal{F}[y_1, \dots, y_n, \dots, y_1^{(h)}, \dots, y_n^{(h)}]$ which lies on ${}^a\mathbb{P}_\sigma^{(k)}, {}^a\mathbb{P}_0^{(l)}$ ($\sigma = 1, \dots, d; k = 0, \dots, h; l = 0, \dots, h - 1$), then there must exist some τ such that $\eta_{j0} = \xi_{\tau j}$ for $j = 1, \dots, n$.

Assume that (\star) is valid. Suppose $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{E}^n$ is a generic point of V_1 which lie on $\mathbb{P}_1, \dots, \mathbb{P}_d$ as well as on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$. Then the algebraic point $(\eta_1, \dots, \eta_n, \dots, \eta_1^{(h)}, \dots, \eta_n^{(h)})$ is a generic point of the algebraic ideal $\mathbb{I}(V_1)^{(h)} = \mathbb{I}(V_1) \cap \mathcal{F}[y_1, \dots, y_n, \dots, y_1^{(h)}, \dots, y_n^{(h)}]$ which lies on ${}^a\mathbb{P}_\sigma^{(k)}, {}^a\mathbb{P}_0^{(l)}$ ($\sigma = 1, \dots, d; k = 0, \dots, h; l = 0, \dots, h - 1$). By (\star), there must exist some τ such that $\eta_j = \xi_{\tau j}$ for $j = 1, \dots, n$. Since η is a differential point of V_1 , thus $\eta = \xi_\tau$ for some τ .

Now we are going to prove (\star). Similar to the proof of Lemma 5.4, rewrite F as an algebraic polynomial $f(u_{\sigma j}^{(k)}, u_{0j}^{(l)}, u_{00}^{(h)}, \dots, u_{0n}^{(h)})$ and consider condition 2 as a pure algebraic factorization. Let $(\eta_{10}, \dots, \eta_{m0}, \dots, \eta_{1h}, \dots, \eta_{nh})$ be such a generic point of $\mathbb{I}(V_1)^{(h)}$ other than $(\xi_{11}, \dots, \xi_{1n}, \dots, \xi_{11}^{(h)}, \dots, \xi_{1n}^{(h)})$. Then we have the \mathcal{F} -isomorphism $\mathcal{F}(\eta_{10}, \dots, \eta_{m0}, \dots, \eta_{1h}, \dots, \eta_{nh}) \cong \mathcal{F}(\xi_{11}, \dots, \xi_{1n}, \dots, \xi_{11}^{(h)}, \dots, \xi_{1n}^{(h)})$, which maps η_{jk} to $\xi_{1j}^{(k)}$ for $j = 1, \dots, n$ and $k = 0, \dots, h$. By Lemma 5.2, there exist $w_{\sigma jk}, w_{0jl} \in \mathcal{E}$ such that the above algebraic isomorphism can be extended to the isomorphism

$$\mathcal{F}(\eta_{10}, \dots, \eta_{m0}, \dots, \eta_{1h}, \dots, \eta_{nh}, u_{\sigma j}^{(k)}, u_{0j}^{(l)}) \cong \mathcal{F}(\xi_{11}, \dots, \xi_{1n}, \dots, \xi_{11}^{(h)}, \dots, \xi_{1n}^{(h)}, w_{\sigma jk}, w_{0jl})$$

where $\sigma = 1, \dots, d; j = 0, \dots, n; k = 0, \dots, h; l = 0, \dots, h - 1$, and $u_{\sigma j}^{(k)}$ and $u_{0j}^{(l)}$ map to $w_{\sigma jk}$ and w_{0jl} , respectively. Since $(\eta_{10}, \dots, \eta_{m0}, \dots, \eta_{1h}, \dots, \eta_{nh})$ lies on ${}^a\mathbb{P}_\sigma^{(k)}, {}^a\mathbb{P}_0^{(l)}$, the relation $u_{\sigma 0}^{(k)} + \sum_{j=1}^n \sum_{m=0}^k \binom{k}{m} u_{\sigma j}^{(m)} \eta_{j, k-m} = 0$ implies that $w_{\sigma 0k} + \sum_{j=1}^n \sum_{m=0}^k \binom{k}{m} w_{\sigma jm} \xi_{1j}^{(k-m)} = 0$ ($\sigma = 1, \dots, d$) and the relation $u_{00}^{(l)} + \sum_{j=1}^n \sum_{m=0}^l \binom{l}{m} u_{0j}^{(m)} \eta_{j, l-m} = 0$ implies that $w_{00l} + \sum_{j=1}^n \sum_{m=0}^l \binom{l}{m} w_{0jm} \xi_{1j}^{(l-m)} = 0$ ($l = 0, \dots, h - 1$). Furthermore, if $w_{00h} + \sum_{i=1}^n w_{0ih} \xi_{1i}^{(h)} + \sum_{j=1}^n \sum_{m=0}^{h-1} \binom{h}{m} w_{0jm} \xi_{1j}^{(h-m)} = 0$ is valid, then from Lemma 5.4 it follows that the algebraic polynomial $f(w_{\sigma jk}, w_{0jl}, w_{00h}, \dots, w_{0nh}) = 0$. Then, by the Hilbert Nullstellensatz, when regarded as a polynomial in the algebraic indeterminates $u_{00}^{(h)}, \dots, u_{0n}^{(h)}$, $f(w_{\sigma jk}, w_{0jl}, u_{00}^{(h)}, \dots, u_{0n}^{(h)}) \in (u_{00}^{(h)} + u_{01}^{(h)} \xi_{11} + \dots + u_{0n}^{(h)} \xi_{1n} + \sum_{j=1}^n \sum_{m=0}^{h-1} \binom{h}{m} w_{0jm} \xi_{1j}^{(h-m)})$. Thus, $u_{00}^{(h)} + u_{01}^{(h)} \xi_{11} + \dots + u_{0n}^{(h)} \xi_{1n} + \sum_{j=1}^n \sum_{m=0}^{h-1} \binom{h}{m} w_{0jm} \xi_{1j}^{(h-m)}$ divides $f(w_{\sigma jk}, w_{0jl}, u_{00}^{(h)}, \dots, u_{0n}^{(h)})$. By the above isomorphism, $f(u_{\sigma j}^{(k)}, u_{0j}^{(l)}, u_{00}^{(h)}, \dots, u_{0n}^{(h)})$ is divisible by $u_{00}^{(h)} + u_{01}^{(h)} \eta_{10} + \dots + u_{0n}^{(h)} \eta_{n0} + \sum_{j=1}^n \sum_{m=0}^{h-1} \binom{h}{m} u_{0j}^{(m)} \eta_{j, h-m}$. The first factorization expression of condition 2 shows that when regarded as an algebraic polynomial in the variables $u_{00}^{(h)}, u_{01}^{(h)}, \dots, u_{0n}^{(h)}, f(u_{\sigma j}^{(k)}, u_{0j}^{(l)}, u_{00}^{(h)}, \dots, u_{0n}^{(h)}) = A \prod_{\tau=1}^g (u_{00}^{(h)} + u_{01}^{(h)} \xi_{\tau 1} + \dots + u_{0n}^{(h)} \xi_{\tau n} + \sum_{j=1}^n \sum_{k=0}^{h-1} \binom{h}{k} u_{0j}^{(k)} \xi_{\tau j}^{(h-k)})$. Thus, there exist some τ such that $\eta_{j0} = \xi_{\tau j}$ ($j = 1, \dots, n$), which completes the proof of the claim.

Denote the dimension and order of V_τ by d_τ and h_τ , respectively. We claim that $d_\tau = d$ and $h_\tau = h$. Since V_τ meets $\mathbb{P}_1, \dots, \mathbb{P}_d$ and $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ are such points in their intersection variety, by Theorem 3.7, $d_\tau \geq d$. If $d_\tau > d \geq 0$, then V_τ meets

$\mathbb{P}_1, \dots, \mathbb{P}_d, \mathbb{P}_0$. Let (η_1, \dots, η_n) be a generic point of $\mathbb{V}(\mathbb{I}(V_\tau), \mathbb{P}_1, \dots, \mathbb{P}_d, \mathbb{P}_0)$. Then by Lemma 5.3, (η_1, \dots, η_n) is also a generic point of V_τ . Since (η_1, \dots, η_n) lies on \mathbb{P}_0 , it also lies on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$. From the above claim, there exist some τ such that $(\eta_1, \dots, \eta_n) = (\xi_{\tau 1}, \dots, \xi_{\tau n})$. Thus, $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ lies on \mathbb{P}_0 , which implies that $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$ is a zero δ -polynomial, which is a contradiction. So $d_\tau = d$.

It remains to show that $h_\tau = h$. We first prove $h_\tau \geq h$. Suppose the contrary; then $h_\tau \leq h - 1$. Similar to the proof of Theorem 4.36, we can prove that $\mathbb{V}([\mathbb{I}(V_\tau), \mathbb{P}_1, \dots, \mathbb{P}_d]^{(h)}, {}^a\mathbb{P}_0, \dots, {}^a\mathbb{P}_0^{(h-1)}) = \emptyset$. But $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ is an element of $\mathbb{V}(\mathbb{I}(V_\tau), \mathbb{P}_1, \dots, \mathbb{P}_d)$ which also lies on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$, which is a contradiction. Now suppose that $h_\tau > h$; then $h_\tau - 1 \geq h$. From Theorems 4.36 and 4.34, every point of V_τ which lies both on $\mathbb{P}_1, \dots, \mathbb{P}_d$ and on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h_\tau-1)}$ is a generic point of V_τ . But the generic points of V_τ which lie on $\mathbb{P}_1, \dots, \mathbb{P}_d$ as well as on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$ are included in $\{(\xi_{\tau 1}, \dots, \xi_{\tau n}) : \tau = 1, \dots, g\}$. So some $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ lies on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h_\tau-1)}$. Since $h_\tau - 1 \geq h$, we have that $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ lies on ${}^a\mathbb{P}_0^{(h)}$, which implies $F(\mathbf{u}_0, \dots, \mathbf{u}_d) \equiv 0$, a contradiction. Thus, we have proved that $d_\tau = d$ and $h_\tau = h$.

Since the solutions of V_τ and $\mathbb{P}_1, \dots, \mathbb{P}_d$ which also lie on ${}^a\mathbb{P}_0, {}^a\mathbb{P}'_0, \dots, {}^a\mathbb{P}_0^{(h-1)}$ are generic points of V_τ , these are therefore contained in $\{(\xi_{\tau 1}, \dots, \xi_{\tau n}) : \tau = 1, \dots, g\}$. Hence, the differential Chow form of V_τ is of the form

$$F_\tau(\mathbf{u}_0, \dots, \mathbf{u}_d) = A_\tau \prod_{\rho=1}^g ((u_{00} + u_{01}\xi_{\tau 1} + \dots + u_{0n}\xi_{\tau n})^{(h)})^{l_{\tau\rho}},$$

where $l_{\tau\rho} = 1$ or 0 according to whether $(\xi_{\tau 1}, \dots, \xi_{\tau n})$ is in V_τ . Since both F_τ and F are irreducible, they differ at most by a factor in \mathcal{F} . Therefore, V_τ ($\tau = 1, \dots, g$) are the same δ -variety, and $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$ is their differential Chow form. \square

In order to define δ -Chow varieties in the next subsection, we will introduce the concept of order-unmixed δ -varieties. A δ -variety V is called *order-unmixed* if all its components have the same dimension and order. Let V be an order-unmixed δ -variety with dimension d and order h and $V = \bigcup_{i=1}^l V_i$ its minimal irreducible decomposition with $F_i(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ the Chow form of V_i . Let

$$(5.1) \quad F(\mathbf{u}_0, \dots, \mathbf{u}_d) = \prod_{i=1}^l F_i(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)^{s_i}$$

with s_i arbitrary nonnegative integers. Associated with (5.1), we introduce the concept of a *differential algebraic cycle*, or simply a δ -cycle, with $\mathbf{V} = \sum_{i=1}^l s_i \mathbf{V}_i$ as a generalization of the concept of an algebraic cycle in algebraic geometry [13], where s_i is called the multiplicity of \mathbf{V}_i in \mathbf{V} . Recall that we have defined the δ -degree m and leading δ -degree g for an irreducible δ -variety V in Definitions 4.19 and 4.33, respectively. Let g_i and m_i be the leading δ -degree and δ -degree of V_i respectively. Then the leading δ -degree and δ -degree of \mathbf{V} is defined to be $\sum_{i=1}^l s_i g_i$ and $\sum_{i=1}^l s_i m_i$, respectively.

Given a δ -polynomial $G(\mathbf{u}_0, \dots, \mathbf{u}_d)$ with $\text{ord}(G, u_{00}) = h$, it may be reducible over \mathcal{F} such that some of its irreducible factors are free of $u_{00}^{(h)}$. In that case, if the product of all such factors is L , then we define the primitive part of G w.r.t. $u_{00}^{(h)}$

to be G/L . Otherwise, its primitive part w.r.t. $u_{00}^{(h)}$ by convention is defined to be itself. Then we have

Theorem 5.5. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ be a δ -polynomial and \tilde{F} the primitive part of F with respect to the variable $u_{00}^{(h)}$. If F satisfies the four conditions in Theorem 5.1, then \tilde{F} is the Chow form for the δ -cycle of dimension d and order h .*

Proof. By definition, $F = B\tilde{F}$, where $\text{ord}(B, \mathbf{u}_0) < h$. Since F is δ -homogenous in \mathbf{u}_i for each i , \tilde{F} is δ -homogenous in each \mathbf{u}_i , too. Also, since B is free of $u_{00}^{(h)}$, i.e. B divides A , then \tilde{F} satisfies conditions 2 and 3, and moreover the $(\xi_{\tau_1}, \dots, \xi_{\tau_n})$ in the factorization are the same as that of F . By the proof of Theorem 5.1, we have that $\mathbb{I}(\xi_{\tau_1}, \dots, \xi_{\tau_n})$ is of dimension d and order h over \mathcal{F} . Then similar to the proof of Lemma 4.1 and Theorem 4.11, we conclude that $\mathbb{I}(-\sum_{j=1}^n v_{0j}\xi_{\tau_j}, \dots, -\sum_{j=1}^n v_{dj}\xi_{\tau_j})$ is of dimension d over $\mathcal{F}\langle v_{ij} : i = 0, \dots, d; j = 1, \dots, n \rangle$ and its relative order w.r.t. any parametric set is h , where v_{ij} ($i = 0, \dots, d; j = 1, \dots, n$) are δ -indeterminates over $\mathcal{F}\langle \xi_{\tau_1}, \dots, \xi_{\tau_n} \rangle$. In particular, $\text{tr.deg } \mathcal{F}\langle \zeta_0, \zeta_1, \dots, \zeta_d \rangle / \mathcal{F}\langle \zeta_1, \dots, \zeta_d \rangle = h$, where $\zeta_i = -\sum_{j=1}^n v_{ij}\xi_{\tau_j}$. Thus $B(-\sum_{j=1}^n v_{0j}\xi_{\tau_j}, \dots, -\sum_{j=1}^n v_{dj}\xi_{\tau_j}) \neq 0$. But $F(-\sum_{j=1}^n v_{0j}\xi_{\tau_j}, \dots, -\sum_{j=1}^n v_{dj}\xi_{\tau_j}) = 0$, so $\tilde{F}(-\sum_{j=1}^n v_{0j}\xi_{\tau_j}, \dots, -\sum_{j=1}^n v_{dj}\xi_{\tau_j}) = 0$. It follows that $\tilde{F}(S^0\xi_\tau, \dots, S^d\xi_\tau) = 0$, for if we suppose the contrary, then $B(S^0\xi_\tau, \dots, S^d\xi_\tau) = 0$. But if we specialize s_{jk}^i ($j < k, j > 0$) to 0 and s_{0k}^i ($k > 0$) to $-v_{ik}$, then

$$B(-\sum_{j=1}^n v_{0j}\xi_{\tau_j}, \dots, -\sum_{j=1}^n v_{dj}\xi_{\tau_j}) = 0,$$

which is a contradiction. Thus, \tilde{F} satisfies condition 4.

Now we claim that \tilde{F} is the Chow form of some δ -cycle. Let V_τ ($\tau = 1, \dots, g$) be the irreducible δ - \mathcal{F} -variety with $(\xi_{\tau_1}, \dots, \xi_{\tau_n})$ as its generic point over \mathcal{F} . Following the steps in the proof of Theorem 5.1 exactly, we arrive at the conclusion that the Chow form of V_τ is of the form

$$F_\tau(\mathbf{u}_0, \dots, \mathbf{u}_d) = A_\tau \prod_{\rho=1}^g ((u_{00} + u_{01}\xi_{\tau_1} + \dots + u_{0n}\xi_{\tau_n})^{(h)})^{l_{\tau\rho}},$$

where $l_{\tau\rho} = 1$ or 0 according to whether $(\xi_{\tau_1}, \dots, \xi_{\tau_n})$ is in V_τ . Since each ξ_τ is in at least one of the δ -varieties V_i , the Chow form of $\bigcup_{\tau=1}^g V_\tau$ is of the form $G(\mathbf{u}_0, \dots, \mathbf{u}_d) = \prod_{\tau=1}^g (F_\tau(\mathbf{u}_0, \dots, \mathbf{u}_d))^{s_\tau} = C(\mathbf{u}_0, \dots, \mathbf{u}_d) \prod_{\rho=1}^g ((u_{00} + u_{01}\xi_{\tau_1} + \dots + u_{0n}\xi_{\tau_n})^{(h)})^{\eta_{\tau\rho}}$ with $\eta_{\tau\rho} > 0$. Since \tilde{F} and G have the same factors $(u_{00} + u_{01}\xi_{\tau_1} + \dots + u_{0n}\xi_{\tau_n})^{(h)}$ and the primitive factor of \tilde{F} w.r.t. $u_{00}^{(h)}$ is itself, thus we can find $\eta_{\tau\rho}$ such that $\tilde{F} = G$, which completes the proof. \square

5.2. Differential Chow quasi-variety for a differential algebraic cycle. A δ -cycle V in the n -dimensional δ -affine space with dimension d , order h , leading δ -degree g , and δ -degree m is said to be of index (n, d, h, g, m) . In this section, we will define the δ -Chow quasi-variety in certain cases such that each point in this δ -variety represents a δ -cycle with a given index (n, d, h, g, m) .

For a given index (n, d, h, g, m) , a δ -polynomial $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$ which has unknown coefficients a_λ ($\lambda = 0, \dots, D$) and satisfies the following two conditions is referred to as a δ -polynomial with index (n, d, h, g, m) .

1) F is a homogenous polynomial of the same degree m in each set of indeterminates $\mathbf{u}_i = (u_{i0}, u_{i1}, \dots, u_{in})$ ($i = 0, \dots, d$) and their derivatives. Furthermore, for each u_{ij} , $\text{ord}(F, u_{ij})$ is either h or $-\infty$. In particular, $\text{ord}(F, u_{00}) = h$.

2) As a polynomial in $u_{00}^{(h)}, u_{01}^{(h)}, \dots, u_{0n}^{(h)}$, its total degree is g . In particular, $\text{deg}(F, u_{00}^{(h)}) = g$.

We want to determine the necessary and sufficient conditions imposed on a_λ ($\lambda = 0, \dots, D$) in order that F be the Chow form for a δ -cycle with index (n, d, h, g, m) . Proceeding in this way, if the necessary and sufficient conditions given in Theorem 5.1 can be expressed by some δ -polynomials in a_λ , then the δ -variety defined by them is called the δ -Chow (quasi)-variety. More precisely, we have the following definition.

Definition 5.6. Let $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$ be a δ -polynomial with δ - \mathcal{F} -indeterminates a_i ($i = 0, \dots, D$) in \mathcal{E} as coefficients and with index (n, d, h, g, m) . A quasi- δ -variety $\mathbb{C}\mathbb{V}$ in the variables a_i is called the *differential Chow quasi-variety* with index (n, d, h, g, m) if a point \bar{a}_i is in $\mathbb{C}\mathbb{V}$ if and only if \tilde{F} is the Chow form for a δ -cycle with index (n, d, h, g, m_1) with $m_1 \leq m$, where \tilde{F} is obtained from F by first replacing a_i by \bar{a}_i and then taking the primitive part with respect to the variable $u_{00}^{(h)}$.

In the case $h = 0$, since Theorems 1.2 and 5.1 become their algebraic counterparts, we can obtain the equations for the algebraic Chow variety in the same way as in [16, pp.56-57]. So in the following, we only consider the case $h > 0$. For $h > 0$, the case $g = 1$ is relatively simple. The following result shows how to determine the defining equations for the δ -Chow quasi-variety with index (n, d, h, g, m) in the case $g = 1$.

Theorem 5.7. Let $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$ be a δ -polynomial with δ - \mathcal{F} -indeterminates a_ν ($\nu = 0, \dots, D$) as coefficients and with index (n, d, h, g, m) with $g = 1$. Let I_F be the initial of F w.r.t. the elimination ranking $u_{00} \succ u_{i,j}$ and a_0, \dots, a_I be the coefficients of I_F . Then we can find a set of δ -homogeneous δ -polynomials

$$R_\omega(a_0, \dots, a_D) \ (\omega = 1, \dots, v)$$

in a_ν such that $\mathbb{V}(R_\omega : \omega = 1, \dots, v) \setminus \mathbb{V}(a_0, \dots, a_I)$ is the δ -Chow quasi-variety of index (n, d, h, g, m) with $g = 1$.

Proof. In order for F to be a differential Chow form, by Theorem 1.2, F must be δ -homogeneous in each \mathbf{u}_i . Let λ be a δ - \mathcal{F} -indeterminate. For each i , replacing \mathbf{u}_i by $\lambda \mathbf{u}_i$ in F , we should have

$$F(\mathbf{u}_0, \dots, \mathbf{u}_{i-1}, \lambda \mathbf{u}_i, \mathbf{u}_{i+1}, \dots, \mathbf{u}_d) = \lambda^m F(\mathbf{u}_0, \dots, \mathbf{u}_d).$$

Comparing the coefficients of the power products of λ, u_{ij} and their derivatives, we obtain a system of linearly homogenous equations $R_\omega(a_0, \dots, a_D) = 0$ ($\omega = 1, \dots, e_1$) in a_ν , which are the conditions for F to be δ -homogeneous and with degree m in each \mathbf{u}_i . So by Gaussian elimination in linear algebra, we can obtain a basis for the solution space of $R_\omega = 0$ ($\omega = 1, \dots, e_1$). More precisely, if the coefficient matrix of this linear equation is of rank r , then r of $\{a_0, \dots, a_D\}$ are the linear combinations of the other $D + 1 - r$ of a_ν . Now substitute these r relations into F and denote the new δ -polynomial by F_1 . That is, F_1 is a δ -homogenous δ -polynomial in each \mathbf{u}_i which only involves $D + 1 - r$ independent coefficients a_ν .

Since $g = 1$, F_1 can be written in the form

$$F_1(\mathbf{u}_0, \dots, \mathbf{u}_d) = A_0 u_{00}^{(h)} + A_1 u_{01}^{(h)} + \dots + A_n u_{0n}^{(h)} + B,$$

where A_i and B are free of $u_{0k}^{(h)}$. Denote $-(A_1 u_{01}^{(h)} + \dots + A_n u_{0n}^{(h)} + B)/A_0$ by γ . Then $u_{00}^{(h)} = \gamma$ is the solution of F_1 as an algebraic polynomial in $u_{00}^{(h)}$. Let $\xi_j = \frac{\partial F_1}{\partial u_{0j}^{(h)}} / \frac{\partial F_1}{\partial u_{00}^{(h)}} \Big|_{u_{00}^{(h)} = \gamma} = A_j/A_0 \Big|_{u_{00}^{(h)} = \gamma}$ for $j = 1, \dots, n$. Proceeding as in the proof of Theorem 4.27, we define the derivatives of ξ_j to be $\xi_j^{(k)} = (\delta \xi_j^{(k-1)}) \Big|_{u_{00}^{(h)} = \gamma}$. It is easy to see that this definition is well defined. Since F_1 is δ -homogeneous in \mathbf{u}_0 , by Theorem 4.16, for $r \neq 0$,

$$\sum_{j=0}^n \sum_{k \geq 0} \binom{k+r}{r} u_{0j}^{(k)} \frac{\partial F_1}{\partial u_{0j}^{(k+r)}} = 0.$$

In the case $r = h$, we have $\sum_{j=0}^n u_{0j} \frac{\partial F_1}{\partial u_{0j}^{(h)}} = 0$. Set $u_{00}^{(h)} = \gamma$ in the identity $\sum_{j=0}^n u_{0j} \frac{\partial F_1}{\partial u_{0j}^{(h)}} = 0$; then we have $u_{00} + \sum_{j=1}^n u_{0j} \xi_j = 0$ with $u_{00}^{(h)} = \gamma$. So $(\xi_1, \dots, \xi_n, \dots, \xi_1^{(h-1)}, \dots, \xi_n^{(h-1)})$ is a solution of ${}^a \mathbb{P}_0, {}^a \mathbb{P}'_0, \dots, {}^a \mathbb{P}_0^{(h-1)}$ and $\gamma = -(\sum_{j=1}^n u_{0j} \xi_j)^{(h)}$. So $F_1(\mathbf{u}_0, \dots, \mathbf{u}_d) = A_0(u_{00} + \sum_{j=1}^n u_{0j} \xi_j)^{(h)}$. As a consequence, with these ξ_i , the second condition and the second part of the third condition in Theorem 5.1 are satisfied.

In order for F_1 to be the Chow form for some δ -variety, by Theorem 5.1, (ξ_1, \dots, ξ_n) should satisfy $\mathbb{P}_\sigma = 0$ ($\sigma = 1, \dots, d$) and $F_1(S^0 \xi, \dots, S^d \xi) = 0$, where S^i are $(n+1) \times (n+1)$ -skew symmetric matrices with elements independent indeterminates and $\xi = (1, \xi_1, \dots, \xi_n)^T$.

First, setting $y_j = A_j/A_0$ in $\mathbb{P}_\sigma = 0$, we get $u_{\sigma 0} A_0 + \sum_{j=1}^n u_{\sigma j} A_j = 0$. Then we obtain some equations in a_ν by equating to zero the coefficients of the various δ -products of $\mathbf{u}_0, \dots, \mathbf{u}_d$. This gives δ -polynomials $R_\omega(a_0, \dots, a_D)$ ($\omega = e_1 + 1, \dots, e_2$).

Second, we obtain some δ -equations $\chi_\tau(a_\nu, y_1, \dots, y_n)$ by equating to zero the coefficients of all δ -products of the independent indeterminates s_{jk}^i ($j > k$) in $F_1(S^0 Y, \dots, S^d Y) = 0$ with $Y = (1, y_1, \dots, y_n)^T$. Then setting $y_j^{(k)} = \xi_j^{(k)}$ in the above χ_τ and clearing denominators, we obtain polynomial equations p_μ in $u_{ij}^{(k)}$ and a_ν . Equating to zero the coefficients of the power products of the $u_{ij}^{(k)}$ in p_μ , we finally obtain δ -polynomials in a_ν : $R_\omega(a_0, \dots, a_D)$ ($\omega = e_2 + 1, \dots, \nu$). We then obtain the defining equations $R_\omega = 0$ for the Chow variety.

We now show that all the R_ω are δ -homogenous δ -polynomials. Denote $\mathbf{a} = (a_0, \dots, a_D)$. We have known for $\omega = 1, \dots, e_1$ that R_ω are linearly homogenous δ -polynomials in \mathbf{a} . Since F_1 as well as A_i are linearly homogenous in \mathbf{a} , $R_\omega(\mathbf{a})$ ($\omega = e_1 + 1, \dots, e_2$) are linearly homogenous δ -polynomials. To show $R_\omega(\mathbf{a})$ ($\omega = e_2 + 1, \dots, \nu$) are δ -homogenous δ -polynomials, by induction on k , we first show that for any δ - \mathcal{F} -indeterminate λ and for each j and k , $\xi_j^{(k)}(\lambda \mathbf{a}) = \xi_j^{(k)}(\mathbf{a})$. By the expression of γ , it is clear that $\gamma(\lambda \mathbf{a}) = \gamma$ since $\xi_j = A_j/A_0 \Big|_{u_{00}^{(h)} = \gamma}$ and $\xi_j(\lambda \mathbf{a}) = \frac{A_j(\lambda \mathbf{a})}{A_0(\lambda \mathbf{a})} \Big|_{u_{00}^{(h)} = \gamma(\lambda \mathbf{a})} = \frac{\lambda A_j(\mathbf{a})}{\lambda A_0(\mathbf{a})} \Big|_{u_{00}^{(h)} = \gamma} = \xi_j(\mathbf{a})$. So it holds for $k = 0$. Suppose it holds for $k - 1$, that is, $\xi_j^{(k-1)}(\lambda \mathbf{a}) = \xi_j^{(k-1)}(\mathbf{a})$. Since $\xi_j^{(k)} = (\delta \xi_j^{(k-1)}) \Big|_{u_{00}^{(h)} = \gamma}$,

$\xi_j^{(k)}(\lambda \mathbf{a}) = \left(\delta(\xi_j^{(k-1)}(\lambda \mathbf{a})) \right) \Big|_{u_{00}^{(h)} = \gamma(\lambda \mathbf{a})} = (\delta \xi_j^{(k-1)}(\mathbf{a})) \Big|_{u_{00}^{(h)} = \gamma(\mathbf{a})} = \xi_j^{(k)}(\mathbf{a})$. Now we are going to show that $R_\omega(\mathbf{a})$ ($\omega = e_2 + 1, \dots, v$) are δ -homogenous δ -polynomials. Since F_1 is linearly homogenous in \mathbf{a} , each $\chi_\tau(a_\nu, y_1, \dots, y_n)$ is linearly homogenous in \mathbf{a} . Setting $y_j^{(k)} = \xi_j^{(k)}$ in χ_τ , it is clear that $\chi_\tau(a_\nu, \xi_1, \dots, \xi_n)(\lambda \mathbf{a}) = \lambda \cdot \chi_\tau(a_\nu, \xi_1, \dots, \xi_n)$ and the denominator of $\chi_\tau(a_\nu, \xi_1, \dots, \xi_n)$ is a pure algebraic homogenous polynomial in \mathbf{a} . Thus, p_μ is δ -homogenous in \mathbf{a} and the δ -homogeneity of R_ω follows.

Let a_0, \dots, a_I be the coefficients of I_F . Then we claim that the quasi-projective δ -variety $\mathbb{C}\mathbb{V} = \mathbb{V}(R_\omega : \omega = 1, \dots, v) \setminus \mathbb{V}(a_0, \dots, a_I)$ is the δ -Chow quasi-variety. Indeed, for every element $(\bar{a}_0, \dots, \bar{a}_D)$ in $\mathbb{C}\mathbb{V}$, following the proof of this theorem, \bar{F} with coefficients \bar{a}_ν satisfies the four conditions in Theorem 5.1. And since $g = 1$, its primitive part must be irreducible and satisfy the four conditions too, which consequently must be the Chow form for some irreducible δ -variety with index $(n, d, h, 1, m_1)$ with $m_1 \leq m$. □

The following example illustrates the procedure to compute the δ -Chow quasi-variety in the case of $g = 1$.

Example 5.8. We consider a δ -polynomial which has 16 terms and has index $(2, 1, 1, 1, 2)$ to illustrate the proof of Theorem 5.7: $F = a_1 u_{12}^2 u_{01} u'_{00} + a_2 u_{11} u_{12} u_{02} u'_{00} + a_3 u_{01} u_{02} u_{12} u'_{10} + a_4 u_{02}^2 u_{11} u'_{10} + a_5 u_{12}^2 u_{00} u'_{01} + a_6 u_{10} u_{12} u_{02} u'_{01} + a_7 u_{00} u_{02} u_{12} u'_{11} + a_8 u_{02}^2 u_{10} u'_{11} + a_9 u_{10} u_{01} u_{12} u'_{02} + a_{10} u_{00} u_{02} u_{11} u'_{12} + a_{11} u_{11} u_{12} u_{00} u'_{02} + a_{12} u_{01} u_{02} u_{10} u'_{12} + a_{13} u_{00} u_{11}^2 u_{02} + a_{14} u_{00} u_{01} u_{11} u_{12} + a_{15} u_{01} u_{10} u_{02} u_{11} + a_{16} u_{10} u_{01}^2 u_{12}$. We will derive the conditions about the coefficients a_ν under which F is a Chow form. First, in order for F to be δ -homogenous, we have $R_1 = a_5 + a_1, R_2 = a_8 + a_4, R_3 = a_9 + a_6, R_4 = a_{10} + a_7, R_5 = a_{11} + a_2, R_6 = a_{12} + a_3$. Replace $a_5, a_8, a_9, a_{10}, a_{11}, a_{12}$ by $-a_1, -a_4, -a_6, -a_7, -a_2, -a_3$ respectively in F to obtain F_1 .

For such an $F_1, A_0 = a_1 u_{12}^2 u_{01} + a_2 u_{11} u_{12} u_{02}, A_1 = -a_1 u_{12}^2 u_{00} + a_6 u_{10} u_{12} u_{02}, A_2 = -a_6 u_{10} u_{01} u_{12} - a_2 u_{11} u_{12} u_{00},$ and $B = a_3 u_{01} u_{02} u_{12} u'_{10} + a_4 u_{02}^2 u_{11} u'_{10} + a_7 u_{00} u_{02} u_{12} u'_{11} - a_4 u_{02}^2 u_{10} u'_{11} - a_7 u_{00} u_{02} u_{11} u'_{12} - a_3 u_{01} u_{02} u_{10} u'_{12} + a_{13} u_{00} u_{11}^2 u_{02} + a_{14} u_{00} u_{01} u_{11} u_{12} + a_{15} u_{01} u_{10} u_{02} u_{11} + a_{16} u_{10} u_{01}^2 u_{12}$. Then $\gamma = (F_1 - A_0 u'_{00})/A_0, \xi_1 = A_1/A_0|_{u'_{00} = \gamma},$ and $\xi_2 = A_2/A_0|_{u'_{00} = \gamma}$. To confirm that $u_{10} + u_{11} \xi_1 + u_{12} \xi_2 = 0$, we must have $R_7 = a_6 - a_1 = 0, R_8 = a_2 + a_1 = 0$.

In order to satisfy the fourth condition of Theorem 5.1, we obtain a set of δ -polynomial equations $R_\omega(a_0, \dots, a_{16}) = 0$ which have more complicated forms. By simplifying them with $R_7 = 0$ and $R_8 = 0$, we obtain $R_9 = a_7 a_1 (a_{15} + a_{16}), R_{10} = a_7 a_1 (a_{14} + a_{16}), R_{11} = a_7 a_1 (a_{13} - a_{16}), R_{12} = a_1 (a_4 - a_7), R_{13} = a_1 (a_3 + a_7), R_{14} = a_1 a_{16} (a_1 - a_7), R_{15} = a_{15} a_1^2 + a_1 a_7 a_{16}, R_{16} = a_{14} a_1^2 + a_1 a_7 a_{16}, R_{17} = a_7 a_1 (a_1 - a_7), R_{18} = a_1^2 a_{13} - a_1 a_7 a_{16}, R_{19} = a_1^3 - a_7^2 a_1$. Thus the Chow quasi-variety is $\mathbb{V}(R_1, \dots, R_{19})/\mathbb{V}(a_1, a_2) = \mathbb{V}(a_2 + a_1, a_3 + a_1, a_4 - a_1, a_5 + a_1, a_6 - a_1, a_7 - a_1, a_8 + a_1, a_9 + a_1, a_{10} + a_1, a_{11} - a_1, a_{12} - a_1, a_{14} + a_{13}, a_{15} + a_{13}, a_{16} - a_{13})/\mathbb{V}(a_1)$. From Example 4.6, it is easy to check that each point of this quasi-variety is the coefficient of the Chow form for $\mathbb{V}(a_1 y'_1 + a_{13} y_2)$ for some $a_1, a_{13} \in \mathcal{F}$. Note that a_{13} could be zero and the result is still valid.

We are unable to prove the existence of the Chow quasi-variety in the case of $g > 1$. The main difficulty is how to perform elimination for a mixed system consisting of both differential and algebraic equations. In our case, condition 2 and the second part of condition 3 of Theorem 5.1 generate algebraic equations

in the coefficients of F and ξ_{ij} , while condition 1, the first part of condition 3, and condition 4 of Theorem 5.1 generate δ -equations. Also, we need to eliminate variables ξ_{ij} from these equations.

The following example shows that the δ -Chow quasi-variety can be easily defined in the case of $n = 1$.

Example 5.9. If $n = 1$, then $d = 0$, and every irreducible δ -homogeneous δ -polynomial in $\mathbf{u}_0 = (u_{00}, u_{01})$ is the differential Chow form for some irreducible δ -variety.

Proof. Let $F(\mathbf{u}_0) = F(u_{00}, u_{01})$ be an irreducible δ -homogenous δ -polynomial with degree m and order h . Then $F(-\frac{u_{00}}{u_{01}}, -1) = (-\frac{1}{u_{01}})^m F(u_{00}, u_{01})$. Let $g(-\frac{u_{00}}{u_{01}}) = F(-\frac{u_{00}}{u_{01}}, -1)$. It is easy to show that $g(y)$ is an irreducible δ -polynomial. By Example 4.6, the Chow form of the prime δ -ideal $\text{sat}(g(y))$ is $(-u_{01})^m g(-\frac{u_{00}}{u_{01}}) = (-u_{01})^m F(-\frac{u_{00}}{u_{01}}, -1) = F(u_{00}, u_{01})$, and the result is proved. \square

As a consequence, the Chow quasi-variety in the case of $n = 1$ always exists.

Example 5.10. Let $\mathbf{u}_0 = (u_{00}, u_{01})$ and $F(\mathbf{u}_0)$ be a homogenous δ -polynomial with index $(1, 0, h, g, m)$ and coefficients a_0, \dots, a_D . Let I_F be the initial of F w.r.t. the elimination ranking $u_{00} \succ u_{01}$ and a_0, \dots, a_I the coefficients of I_F . Let $R_\omega(a_0, \dots, a_D) (\omega = 1, \dots, e_1)$ be the equations obtained in the proof of Theorem 5.7. Then under the condition $R_\omega(a_0, \dots, a_D) = 0 (\omega = 1, \dots, e_1)$, F will become a δ -homogenous δ -polynomial. Then by Example 5.9, $\mathbb{V}(R_\omega : \omega = 1, \dots, e_1) \setminus \mathbb{V}(a_0, \dots, a_I)$ is the δ -Chow quasi-variety with index $(1, 0, h, g, m)$.

6. GENERALIZED DIFFERENTIAL CHOW FORM AND DIFFERENTIAL RESULTANT

We mentioned that the differential Chow form can be obtained by intersecting the δ -variety with generic δ -hyperplanes. In this section, we show that when intersecting an irreducible δ -variety of dimension d by $d + 1$ generic δ -hypersurfaces, we can obtain the generalized Chow form which has similar properties to the Chow form. As a direct consequence, we can define the differential resultant and obtain its properties.

6.1. Generalized differential Chow form. Let $V \subset \mathcal{E}^n$ be an irreducible δ - \mathcal{F} -variety with dimension d and order h , and

$$(6.1) \quad \mathbb{P}_i = u_{i0} + \sum_{j=1}^n \sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_i+1)} \\ 1 < |\alpha| \leq m_i}} u_{i\alpha} (\mathbb{Y}^{[s_i]})^\alpha \quad (i = 0, \dots, d)$$

be a generic δ -polynomial of order $s_i \geq 0$ and degree $m_i \geq 1$, where $u_{i0}, u_{ijk}, u_{i\alpha}$ ($i = 0, \dots, d; j = 1, \dots, n; k = 0, \dots, s_i; \alpha \in \mathbb{Z}_{\geq 0}^{n(s_i+1)}, 1 < |\alpha| \leq m_i$) are δ - \mathcal{F} -indeterminates in \mathcal{E} and $(\mathbb{Y}^{[s_i]})^\alpha$ is a monomial in $\mathcal{F}[\mathbb{Y}^{[s_i]}]$ with exponent vector $\alpha = (\alpha_{10}, \dots, \alpha_{n0}, \alpha_{11}, \dots, \alpha_{n1}, \dots, \alpha_{1s_1}, \dots, \alpha_{ns_1})$, i.e. $(\mathbb{Y}^{[s_i]})^\alpha = \prod_{j=1}^n \prod_{k=0}^{s_i} (y_j^{(k)})^{\alpha_{jk}}$

and $|\alpha| = \sum_{j=1}^n \sum_{k=0}^{s_i} \alpha_{jk}$. For convenience in the rest of the paper, we denote the nonlinear part of each \mathbb{P}_i by f_i , that is,

$$\mathbb{P}_i = u_{i0} + \sum_{j=1}^n \sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} + f_i \quad (i = 0, \dots, d).$$

Denote \mathbf{u} to be the set consisting of all the u_{ijk} and $u_{i\alpha}$ for $i = 0, \dots, d$. Let $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{E}^n$ be a generic point of V , which is free from $\mathcal{F}\langle \mathbf{u}, u_{00}, \dots, u_{d0} \rangle$. We define $d + 1$ elements $\zeta_0, \zeta_1, \dots, \zeta_d$ of $\mathcal{F}\langle \mathbf{u}, \xi_1, \dots, \xi_n \rangle$:

$$(6.2) \quad \zeta_i = - \sum_{j=1}^n \sum_{k=0}^{s_i} u_{ijk} \xi_j^{(k)} - f_i(\xi_1, \dots, \xi_n) \quad (i = 0, \dots, d).$$

Similar to the proof of Lemma 4.1, we can prove that if $d > 0$, then any set of d elements of ζ_0, \dots, ζ_d is a δ -transcendence basis of $\mathcal{F}\langle \mathbf{u} \rangle \langle \zeta_0, \dots, \zeta_d \rangle$ over $\mathcal{F}\langle \mathbf{u} \rangle$. If $d = 0$, ζ_0 is δ -algebraic over $\mathcal{F}\langle \mathbf{u} \rangle$. We thus have

Lemma 6.1. $\text{d.tr.deg } \mathcal{F}\langle \mathbf{u} \rangle \langle \zeta_0, \dots, \zeta_d \rangle / \mathcal{F}\langle \mathbf{u} \rangle = d$.

Let \mathbb{I}_ζ be the prime δ -ideal in $\mathcal{R} = \mathcal{F}\langle \mathbf{u} \rangle \{z_0, \dots, z_d\}$ having $\zeta = (\zeta_0, \dots, \zeta_d)$ as a generic point. By Lemma 6.1, the dimension of \mathbb{I}_ζ is d . Then, the characteristic set of \mathbb{I}_ζ w.r.t. any ranking consists of an irreducible δ -polynomial $g(z_0, \dots, z_d)$ in \mathcal{R} and $\mathbb{I}_\zeta = \text{sat}(g)$. Since the coefficients of $g(z_0, \dots, z_d)$ are elements in $\mathcal{F}\langle \mathbf{u} \rangle$, without loss of generality we assume that $g(\mathbf{u}; z_0, \dots, z_d)$ is irreducible in $\mathcal{F}\langle \mathbf{u}; z_0, \dots, z_d \rangle$. We shall subsequently replace z_0, \dots, z_d by u_{00}, \dots, u_{d0} , and obtain

$$(6.3) \quad G(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = g(\mathbf{u}; u_{00}, \dots, u_{d0}),$$

where $\mathbf{u}_i = (u_{i0}, \dots, u_{ijk}, \dots, u_{i\alpha}, \dots)$ is the sequence of the coefficients of \mathbb{P}_i .

Definition 6.2. The δ -polynomial defined in (6.3) is called the *generalized Chow form* of V or the prime δ -ideal $\mathbb{I}(V)$ with respect to $\mathbb{P}_i (i = 0, \dots, d)$.

Similar to Theorem 4.17, we can prove that the generalized Chow form is a δ -homogeneous δ -polynomial in each set of indeterminates \mathbf{u}_i , but in this case the homogeneous degree for distinct \mathbf{u}_i may be distinct. The *order of the generalized Chow form w.r.t. \mathbf{u}_i* , denoted by $\text{ord}(G, \mathbf{u}_i)$, is defined to be $\max_{u \in \mathbf{u}_i} \text{ord}(g, u)$. Now we will consider the order of the generalized Chow form.

Theorem 6.3. *Let \mathcal{I} be a prime δ -ideal with dimension d and order h defined over \mathcal{F} , and $G(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = g(\mathbf{u}; u_{00}, u_{10}, \dots, u_{d0})$ be its generalized Chow form. Then for a fixed i between 0 and d , $\text{ord}(g, u_{i0}) = h + s - s_i$ with $s = \sum_{l=0}^d s_l$. Moreover, $\text{ord}(G, \mathbf{u}_i) = h + s - s_i$.*

Proof. Use the notation as above in this section. Let $\mathcal{I}_d = [\mathcal{I}, \mathbb{P}_0, \dots, \mathbb{P}_{i-1}, \mathbb{P}_{i+1}, \dots, \mathbb{P}_d] \subset \mathcal{F}\langle \mathbf{u}_0, \dots, \mathbf{u}_{i-1}, \mathbf{u}_{i+1}, \dots, \mathbf{u}_d \rangle \{y_1, \dots, y_n\}$. By Theorem 3.13, \mathcal{I}_d is a prime δ -ideal with dimension 0 and order $h + s_0 + \dots + s_{i-1} + s_{i+1} + \dots + s_d = h + s - s_i$, where $s = \sum_{l=0}^d s_l$.

Let $\mathbb{I}_{\zeta, \xi} = [\mathcal{I}, \mathbb{P}_0, \dots, \mathbb{P}_d] \subset \mathcal{F}\langle \mathbf{u} \rangle \{u_{00}, \dots, u_{d0}, y_1, \dots, y_n\}$ and $\mathcal{I}_1 = [\mathcal{I}, \mathbb{P}_0, \dots, \mathbb{P}_d] = [\mathcal{I}_d, \mathbb{P}_0] \subset \mathcal{F}\langle \widehat{\mathbf{u}} \setminus \{u_{i0}\} \rangle \{u_{i0}, y_1, \dots, y_n\}$, where $\widehat{\mathbf{u}} = \mathbf{u}_0 \cup \dots \cup \mathbf{u}_d$. Denote $\text{ord}(G, u_{i0})$ by h_1 . Similar to the proof of Lemma 4.10, we can show that $\mathcal{A} = g(\mathbf{u}; u_{00}, \dots, u_{d0}), \frac{\partial g}{\partial u_{i0}^{(h_1)}} y_1 - \frac{\partial g}{\partial u_{i10}^{(h_1)}}, \dots, \frac{\partial g}{\partial u_{i0}^{(h_1)}} y_n - \frac{\partial g}{\partial u_{in0}^{(h_1)}}$ is a characteristic set of $\mathbb{I}_{\zeta, \xi}$ w.r.t. the elimination ranking $u_{00} \prec \dots \prec u_{i-1,0} \prec u_{i+1,0} \prec u_{d0} \prec u_{i0} \prec y_1 \prec \dots \prec y_n$. Clearly, \mathcal{I}_1 is the δ -ideal generated by $\mathbb{I}_{\zeta, \xi}$ in $\mathcal{F}\langle \widehat{\mathbf{u}} \setminus \{u_{i0}\} \rangle \{u_{i0}, y_1, \dots, y_n\}$. Since $\{u_{00}, \dots, u_{i-1,0}, u_{i+1,0}, \dots, u_{d0}\}$ is a parametric set of $\mathbb{I}_{\zeta, \xi}$, \mathcal{A} is also a characteristic set of \mathcal{I}_1 w.r.t. the elimination ranking $u_{i0} \prec y_1 \prec \dots \prec y_n$. Since $\text{dim}(\mathcal{I}_1) = 0$, from Corollary 2.10 we have $\text{ord}(\mathcal{I}_1) = \text{ord}(\mathcal{A}) = \text{ord}(g, u_{i0})$.

On the other hand, let $\mathcal{I}_1^{(l)} = [\mathcal{I}_d, u_{i0}^{(l)} + \sum_{j=1}^n \sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} + f_i] \subset \mathcal{F}\langle \widehat{\mathbf{u}} \setminus \{u_{i0}\} \rangle \{u_{i0}, y_1, \dots, y_n\}$ ($l = 1, \dots, s_i$). Since $\text{dim}(\mathcal{I}_1^{(l)}) = 0$, u_{i0} is a leading variable

of $\mathcal{I}_1^{(l)}$ for any ranking. Thus, by Lemma 3.11, we have $\text{ord}(\mathcal{I}_1^{(l+1)}) = \text{ord}(\mathcal{I}_1^{(l)}) + 1$, which follows that $\text{ord}(\mathcal{I}_1^{(s_i)}) = \text{ord}(\mathcal{I}_1) + s_i$, and it is easy to see that $\text{ord}(\mathcal{I}_1^{(s_i)}) = \text{ord}(\mathcal{I}_d) + s_i$. Indeed, let \mathcal{A} be a characteristic set of \mathcal{I}_d w.r.t. some orderly ranking \mathcal{R} , and let t be the pseudo remainder of $u_{i0}^{(s_i)} + \sum_{j=1}^n \sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} + f_i$ w.r.t. \mathcal{A} under the ranking \mathcal{R} . Clearly, $\text{ord}(t, u_{i0}) = s_i$. It is obvious that for some orderly ranking, $\{\mathcal{A}, t\}$ is a characteristic set of $\mathcal{I}_1^{(s_i)}$ with $\text{ld}(\mathcal{A})$ and $u_{i0}^{(s_i)}$ as its leaders, so $\text{ord}(\mathcal{I}_1^{(s_i)}) = \text{ord}(\mathcal{I}_d) + s_i$. Thus, $\text{ord}(\mathcal{I}_1) = \text{ord}(\mathcal{I}_d) = h + s - s_i$, and consequently, $\text{ord}(g, u_{i0}) = h + s - s_i$.

It remains to show that $\text{ord}(g, u_{ijk})$ ($j = 1, \dots, n; k = 0, 1, \dots, s_i$) and $\text{ord}(g, u_{i\alpha})$ cannot exceed $\text{ord}(g, u_{i0})$. If $\text{ord}(g, u_{ijk}) = l > \text{ord}(g, u_{i0})$, then differentiating the identity $g(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$ w.r.t. $u_{ijk}^{(l)}$, we have $\frac{\partial g}{\partial u_{ijk}^{(l)}}(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$. Thus, $\frac{\partial g}{\partial u_{ijk}^{(l)}}$ can be divisible by g , a contradiction. So $\text{ord}(g, u_{ijk}) \leq \text{ord}(g, u_{i0})$. Similarly, we can prove that $\text{ord}(g, u_{i\alpha}) \leq \text{ord}(g, u_{i0})$. Thus, $\text{ord}(G, \mathbf{u}) = \text{ord}(g, u_{i0})$. \square

In the following, we consider the factorization of the generalized Chow form. Denote $h + s - s_i$ by h_i ($i = 0, \dots, d$), where $s = \sum_{l=0}^d s_l$. Now consider G as a polynomial in $u_{00}^{(h_0)}$ with coefficients in $\mathcal{F}_0 = \mathcal{F}(\tilde{\mathbf{u}})(u_{00}, \dots, u_{00}^{(h_0-1)})$, where $\tilde{\mathbf{u}} = \bigcup_{i=0}^d \mathbf{u}_i \setminus \{u_{00}\}$. Then, in an algebraic extension field of \mathcal{F}_0 we have

$$g = A \prod_{\tau=1}^{t_0} (u_{00}^{(h_0)} - \gamma_\tau),$$

where $t_0 = \text{deg}(g, u_{00}^{(h_0)})$. Let $\xi_{\tau\rho k} = g_{\tau\rho k} / g_{\tau 0}$ ($\rho = 1, \dots, n; k = 0, \dots, s_0$) and $\xi_{\tau\alpha} = g_{\tau\alpha} / g_{\tau 0}$, where $g_{\tau\rho k} = \frac{\partial g}{\partial u_{0\rho k}^{(h_0)}} \Big|_{u_{00}^{(h_0)} = \gamma_\tau}$, $g_{\tau\alpha} = \frac{\partial g}{\partial u_{0\alpha}^{(h_0)}} \Big|_{u_{00}^{(h_0)} = \gamma_\tau}$ and $g_{\tau 0} = \frac{\partial g}{\partial u_{00}^{(h_0)}} \Big|_{u_{00}^{(h_0)} = \gamma_\tau}$. Similar to Section 4.4, we can uniquely define the derivatives of γ_τ and $\xi_{\tau\rho 0}$ to make them elements in a differential extension field of $\mathcal{F}(\tilde{\mathbf{u}})$. From $g(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$, if we differentiate this equality w.r.t. $u_{0\rho k}^{(h_0)}$, then we have

$$(6.4) \quad \overline{\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}} + \overline{\frac{\partial g}{\partial u_{00}^{(h_0)}}}(-\xi_\rho^{(k)}) = 0,$$

and if we differentiate $g(\mathbf{u}; \zeta_0, \dots, \zeta_d) = 0$ w.r.t. $u_{0\alpha}^{(h_0)}$, then

$$(6.5) \quad \overline{\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}} + \overline{\frac{\partial g}{\partial u_{00}^{(h_0)}}}(-(\xi^{(s_0)})^\alpha) = 0,$$

where $(\xi^{(s_0)})^\alpha = (\mathbb{Y}^{(s_0)})^\alpha|_{(y_1, \dots, y_n) = (\xi_1, \dots, \xi_n)}$. In the above equations, $\overline{\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}}$ and $\overline{\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}}$ represent $\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}$ and $\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}$ when substituting u_{i0} by ζ_i . For each $\rho = 1, \dots, n$ and $k = 0, \dots, s_0$, multiplying the equations in (6.4) by $u_{0\rho k}$, and for $\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)}$, $1 < |\alpha| \leq m_0$, multiplying the equations in (6.5) by $u_{0\alpha}$ and then

adding all of the equations obtained together, we have

$$\zeta_0 \frac{\overline{\partial g}}{\partial u_{00}^{(h_0)}} + \sum_{\rho=1}^n \sum_{k=0}^{s_0} u_{0\rho k} \frac{\overline{\partial g}}{\partial u_{0\rho k}^{(h_0)}} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1 < |\alpha| \leq m_0}} u_{0\alpha} \frac{\overline{\partial g}}{\partial u_{0\alpha}^{(h_0)}} = 0.$$

Thus, the δ -polynomial $u_{00} \frac{\partial g}{\partial u_{00}^{(h_0)}} + \sum_{\rho=1}^n \sum_{k=0}^{s_0} u_{0\rho k} \frac{\partial g}{\partial u_{0\rho k}^{(h_0)}} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1 < |\alpha| \leq m_0}} u_{0\alpha} \frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}$ van-

ishes at $(u_{00}, \dots, u_{d0}) = (\zeta_0, \dots, \zeta_d)$. Since it is at most of the same order as g , it must be divisible by g . Also, since it has the same degree as g , there exists some $a \in \mathcal{F}$ such that

$$u_{00} \frac{\partial g}{\partial u_{00}^{(h_0)}} + \sum_{\rho=1}^n \sum_{k=0}^{s_0} u_{0\rho k} \frac{\partial g}{\partial u_{0\rho k}^{(h_0)}} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1 < |\alpha| \leq m_0}} u_{0\alpha} \frac{\partial g}{\partial u_{0\alpha}^{(h_0)}} = ag.$$

Setting $u_{00}^{(h_0)} = \gamma_\tau$ in both sides of the above equation, we have

$$u_{00} g_{\tau 0} + \sum_{\rho=1}^n \sum_{k=0}^{s_0} u_{0\rho k} g_{\tau \rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1 < |\alpha| \leq m_0}} u_{0\alpha} g_{\tau \alpha} = 0$$

or

$$u_{00} + \sum_{\rho=1}^n \sum_{k=0}^{s_0} u_{0\rho k} \xi_{\tau \rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1 < |\alpha| \leq m_0}} u_{0\alpha} \xi_{\tau \alpha} = 0.$$

Then, we have

$$\begin{aligned} & (u_{00} + \sum_{\rho=1}^n \sum_{k=0}^{s_0} u_{0\rho k} \xi_{\tau \rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1 < |\alpha| \leq m_0}} u_{0\alpha} \xi_{\tau \alpha})^{(h_0)} \\ &= \gamma_\tau + \left(\sum_{\rho=1}^n \sum_{k=0}^{s_0} u_{0\rho k} \xi_{\tau \rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1 < |\alpha| \leq m_0}} u_{0\alpha} \xi_{\tau \alpha} \right)^{(h_0)} = 0. \end{aligned}$$

We have the following theorem.

Theorem 6.4. *Let $G(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ be the generalized Chow form of a δ - \mathcal{F} -variety of dimension d and order h . Then, there exist $\xi_{\tau\rho}$ ($\rho = 1, \dots, n; \tau = 1, \dots, t_0$) in a δ -extension field of $\mathcal{F}(\tilde{\mathbf{u}})$ such that*

$$(6.6) \quad G(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) \prod_{\tau=1}^{t_0} \mathbb{P}_0(\xi_{\tau 1}, \dots, \xi_{\tau n})^{(h_0)},$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$ is in $\mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d\}$ and $t_0 = \deg(G, u_{00}^{(h_0)})$.

Proof. From what we have proved,

$$G(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) = A(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d) \prod_{\tau=1}^{t_0} \left(u_{00} + \sum_{\rho=1}^n \sum_{k=0}^{s_0} u_{0\rho k} \xi_{\tau\rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1 < |\alpha| \leq m_0}} u_{0\alpha} \xi_{\tau\alpha} \right)^{(h_0)}.$$

Denote $\xi_{\tau\rho 0}$ by $\xi_{\tau\rho}$. To complete the proof, it remains to show that $\xi_{\tau\rho k} = (\xi_{\tau\rho 0})^{(k)} (k = 1, \dots, s_0)$ and $\xi_{\tau\alpha} = \prod_{\rho=1}^n \prod_{j=0}^{s_0} \left((\xi_{\tau\rho 0})^{(j)} \right)^{\alpha_{\rho j}}$. From equation (6.4) and equation (6.5), we have $\xi_{\rho}^{(k)} = \frac{\partial q}{\partial u_{0\rho k}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} (k = 0, \dots, s_0)$ and $(\xi^{(s_0)})^\alpha = \frac{\partial q}{\partial u_{0\alpha}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}}$. So we have the equalities $\left(\frac{\partial q}{\partial u_{0\rho k}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} \right)^{(k)} = \frac{\partial q}{\partial u_{0\rho k}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}}$ and $\frac{\partial q}{\partial u_{0\alpha}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} = \prod_{\rho=1}^n \prod_{j=0}^{s_0} \left(\left(\frac{\partial q}{\partial u_{0\rho 0}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} \right)^{(j)} \right)^{\alpha_{\rho j}}$. Thus, $\left(\frac{\partial q}{\partial u_{0\rho k}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} \right)^{(k)} - \frac{\partial q}{\partial u_{0\rho k}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}}$ and $\frac{\partial q}{\partial u_{0\alpha}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} - \prod_{\rho=1}^n \prod_{j=0}^{s_0} \left(\left(\frac{\partial q}{\partial u_{0\rho 0}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} \right)^{(j)} \right)^{\alpha_{\rho j}}$ vanish at $(u_{00}, \dots, u_{d0}) = (\zeta_0, \dots, \zeta_d)$. Similar to the proof of Theorem 4.34, we can see that both of the differential polynomials $\left(\frac{\partial q}{\partial u_{0\rho 0}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} \right)^{(k)} - \frac{\partial q}{\partial u_{0\rho k}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}}$ and $\frac{\partial q}{\partial u_{0\alpha}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} - \prod_{\rho=1}^n \prod_{j=0}^{s_0} \left(\left(\frac{\partial q}{\partial u_{0\rho 0}^{(h_0)}} / \frac{\partial q}{\partial u_{00}^{(h_0)}} \right)^{(j)} \right)^{\alpha_{\rho j}}$ vanish at $u_{00}^{(h_0+j)} = \gamma_\tau^{(j)} (j \geq 0)$. Thus, $\xi_{\tau\rho 0}^{(k)} = \xi_{\tau\rho k}$ and $\xi_{\tau\alpha} - \prod_{\rho=1}^n \prod_{j=0}^{s_0} \left((\xi_{\tau\rho 0})^{(j)} \right)^{\alpha_{\rho j}} = 0$. The proof is completed. \square

Theorem 6.5. *The points $(\xi_{\tau 1}, \dots, \xi_{\tau n}) (\tau = 1, \dots, t_0)$ in (6.6) are generic points of the δ - \mathcal{F} -variety V and satisfy the equations*

$$\mathbb{P}_\sigma(y_1, \dots, y_n) = u_{\sigma 0} + \sum_{\rho=1}^n \sum_{k=0}^{s_\sigma} u_{\sigma\rho k} y_\rho^{(k)} + f_\sigma = 0 \quad (\sigma = 1, \dots, d).$$

Proof. The proof is similar to that of Theorem 4.34. \square

Theorem 6.6. *Let $G(\mathbf{u}_0, \dots, \mathbf{u}_d)$ be the generalized Chow form of V and $S_G = \frac{\partial G}{\partial u_{00}^{(h_0)}}$ with $\text{ord}(G, u_{00}) = h_0$. Suppose that $\mathbf{u}_i (i = 0, \dots, d)$ are δ -specialized to sets \mathbf{v}_i of specific elements in \mathcal{E} and $\overline{\mathbb{P}}_i (i = 0, \dots, d)$ are obtained by substituting \mathbf{u}_i by \mathbf{v}_i in \mathbb{P}_i . If $\overline{\mathbb{P}}_i = 0 (i = 0, \dots, d)$ meet V , then $G(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$. Furthermore, if $G(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$ and $S_G(\mathbf{v}_0, \dots, \mathbf{v}_d) \neq 0$, then the $d + 1$ δ -hypersurfaces $\overline{\mathbb{P}}_i = 0 (i = 0, \dots, d)$ meet V .*

Proof. The proof is similar to that of Theorem 4.43. \square

6.2. Differential resultant of multivariate differential polynomials. As an application of the generalized Chow form, we can define the differential resultant of $n + 1$ generic δ -polynomials in n variables. Let $\mathcal{I} = [0]$ be the δ -ideal generated by 0 in $\mathcal{F}\{\mathbb{Y}\}$. Then $\dim(\mathcal{I}) = n$. Let $G(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n)$ be the generalized Chow form for \mathcal{I} . Then we will define $G(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n)$ to be the differential resultant for the $n + 1$ generic δ -polynomials given in (6.1).

Definition 6.7. The *differential resultant* for the $n + 1$ generic δ -polynomials \mathbb{P}_i in (6.1) is defined to be the generalized Chow form of $\mathcal{I} = [0]$ associated with these \mathbb{P}_i , and will be denoted by $R(\mathbf{u}_0, \dots, \mathbf{u}_n) = G(\mathbf{u}_0, \dots, \mathbf{u}_n)$.

Theorem 6.8. Let $R(\mathbf{u}_0, \dots, \mathbf{u}_n)$ be the differential resultant of the $n + 1$ δ -polynomials $\mathbb{P}_0, \dots, \mathbb{P}_n$ given in (6.1) with $\text{ord}(\mathbb{P}_i) = s_i$ and $\text{deg}(\mathbb{P}_i) = m_i$, where $\mathbf{u}_i = (u_{i0}, \dots, u_{ijk}, \dots, u_{i\alpha}, \dots)$ ($i = 0, \dots, n$). Denote $s = \sum_{i=0}^n s_i$, $D = \max_{i=0}^n \{m_i\}$ and $\mathbf{u} = \bigcup_{i=0}^n \mathbf{u}_i \setminus \{u_{i0}\}$. Then there exist $h_{jk} \in \mathcal{F}(\mathbf{u})[y_1, \dots, y_n, \dots, y_1^{(s)}, \dots, y_n^{(s)}]$ such that

$$R(\mathbf{u}_0, \dots, \mathbf{u}_n) = \sum_{j=0}^n \sum_{k=0}^{s-s_j} h_{jk} \delta^k \mathbb{P}_j.$$

Moreover, the degree of h_{jk} in \mathbb{Y} is bounded by $(sn + n)^2 D^{sn+n} + D(sn + n)$.

Proof. Let \mathcal{J} be the δ -ideal generated by $\mathbb{P}_0, \dots, \mathbb{P}_n$ in $\mathcal{F}\{\mathbf{u}_0, \dots, \mathbf{u}_n, y_1, \dots, y_n\}$. Let \mathcal{R} be the elimination ranking $\mathbf{u} \prec y_n \prec \dots \prec y_1 \prec u_{n0} \prec \dots \prec u_{00}$ with arbitrary ranking endowed on $\Theta(\mathbf{u}) = (\theta u : u \in \mathbf{u}; \theta \in \Theta)$. Clearly, \mathcal{J} is a prime δ -ideal with $\mathbb{P}_0, \dots, \mathbb{P}_n$ as its characteristic set w.r.t. \mathcal{R} . Thus, $\mathbf{u} \cup \{y_1, \dots, y_n\}$ is a parametric set of \mathcal{J} . From the definition of R , $R \in \mathcal{J}$. In $R(\mathbf{u}_0, \dots, \mathbf{u}_n) = G(\mathbf{u}_0, \dots, \mathbf{u}_n) = g(\mathbf{u}; u_{00}, \dots, u_{n0})$, let u_{i0} ($i = 0, \dots, n$) be replaced respectively by

$$\mathbb{P}_i - \sum_{j=1}^n \sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} - \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_i+1)} \\ 1 < |\alpha| \leq m_i}} u_{i\alpha} (\mathbb{Y}^{(s_i)})^\alpha, \quad (i = 0, \dots, d),$$

and let R be expanded as a polynomial in $\mathbb{P}_0, \dots, \mathbb{P}_n$ and their derivatives. The term not involving $\mathbb{P}_0, \dots, \mathbb{P}_n$ or their derivatives will be a δ -polynomial only involving $\mathbf{u} \cup \{y_1, \dots, y_n\}$ which also belong to \mathcal{J} . Since $\mathcal{J} \cap \mathcal{F}\{\mathbf{u}, y_1, \dots, y_n\} = \{0\}$, such a term will be identically zero. So R is a linear combination of $\mathbb{P}_0, \dots, \mathbb{P}_n$ and some of their derivatives. Since $\text{ord}(R, u_{i0}) = s - s_i$, the above expansion for R involves \mathbb{P}_i only up to the order $s - s_i$ and the coefficients in the linear combination are δ -polynomials in $\mathcal{F}\{\mathbf{u}\}[y_1, \dots, y_n, \dots, y_1^{(s)}, \dots, y_n^{(s)}]$. Denote $\mathcal{R} = \mathcal{F}(\mathbf{u})[y_1, \dots, y_n, \dots, y_1^{(s)}, \dots, y_n^{(s)}]$. Thus, $R \in (\delta^{s-s_0} \mathbb{P}_0, \dots, \delta \mathbb{P}_0, \mathbb{P}_0, \dots, \delta^{s-s_n} \mathbb{P}_n, \dots, \delta \mathbb{P}_n, \mathbb{P}_n) \subseteq \mathcal{R}$, which implies that $(\delta^{s-s_0} \mathbb{P}_0, \dots, \delta \mathbb{P}_0, \mathbb{P}_0, \dots, \delta^{s-s_n} \mathbb{P}_n, \dots, \delta \mathbb{P}_n, \mathbb{P}_n)$ in \mathcal{R} is the unit ideal. By [31, Theorem 1], there exist $A_{jk} \in \mathcal{R}$ with $\text{deg}(A_{jk}) \leq (sn + n)^2 D^{sn+n} + D(sn + n)$ such that

$$1 = \sum_{j=0}^n \sum_{k=0}^{s-s_j} A_{jk} \delta^k \mathbb{P}_j,$$

where $D = \max\{m_0, m_1, \dots, m_n\}$. If we multiply the above equation by R and denote $A_{jk} R$ by h_{jk} , we complete the proof. \square

As a consequence of the above five theorems proved in this section, the properties of the differential resultant listed in Theorem 1.3 are proved.

Let $R(\mathbf{u}_0, \dots, \mathbf{u}_n)$ be the differential resultant for the $n + 1$ generic δ -polynomials \mathbb{P}_i in (6.1). When each \mathbf{u}_i are specialized to specific elements $\mathbf{v}_i \in \mathcal{F}^{n+1}$, $R(\mathbf{v}_0, \dots, \mathbf{v}_n)$ is called the *differential resultant* of $\overline{\mathbb{P}}_i$ ($i = 0, \dots, n$) which are obtained by replacing \mathbf{u}_i by \mathbf{v}_i in \mathbb{P}_i . By Theorem 6.8, the vanishing of the differential resultant for $n + 1$ δ -polynomials in $\mathcal{F}\{\mathbb{Y}\}$ is a necessary condition for them to have a common solution.

Remark 6.9. It is easy to see that if $s_i = 0$, then the differential resultant of $\mathbb{P}_i (i = 0, \dots, n)$ becomes the Macaulay resultant for $n+1$ polynomials in n variables [9, 18]. From Theorem 1.3, we see that the differential resultant has similar properties to that of the Macaulay resultant. Special attention should be paid to the second property which is a differential analog to the so-called Poisson type formulas for algebraic resultants [28]. Also note that many properties of the Macaulay resultant are yet to be extended to the differential case. The most significant one might be to find a matrix representation for the differential resultant similar to the one given in [9, p.102]. Note that such a formula was claimed to be given in [5, 6], which is not correct, as we mentioned in Section 1 of this paper. As a latest development, we defined the differential sparse resultant and proposed a single exponential algorithm to compute it [26].

Similar to the differential Chow form, the differential resultant can be computed with the differential elimination algorithms [1, 7, 32, 39, 14].

Example 6.10. The simplest nonlinear differential resultant is the case $n = 1, d_0 = d_1 = 2, s_0 = 0, s_1 = 1$. Denote y_1 by y . Let $\mathbb{P}_0 = u_{00} + u_{01}y + u_{02}y^2$, $\mathbb{P}_1 = u_{10} + u_{11}y + u_{12}y' + u_{13}y^2 + u_{14}yy' + u_{15}(y')^2$. Then the differential resultant for \mathbb{P}_0 and \mathbb{P}_1 is a δ -polynomial $R(\mathbf{u}_0, \mathbf{u}_1)$ such that $\text{ord}(R, \mathbf{u}_0) = 1$, $\text{ord}(R, \mathbf{u}_1) = 0$ and R is δ -homogenous of degree 8 in \mathbf{u}_0 and degree 2 in \mathbf{u}_1 , respectively. Totally, R has 206 terms. Moreover, R has a matrix representation which is a factor of the determinant of the coefficient matrix of $\mathbb{P}_0, y'\mathbb{P}_0, y^2\mathbb{P}_0, yy'\mathbb{P}_0, y'^2\mathbb{P}_0, \mathbb{P}'_0, y\mathbb{P}'_0, y'\mathbb{P}'_0, yy'\mathbb{P}'_0, y'^2\mathbb{P}'_0, \mathbb{P}_1, y\mathbb{P}_1, y'\mathbb{P}_1, yy'\mathbb{P}_1$ w.r.t. the monomials $\{y^{l_0}(y')^{l_1} | 0 \leq l_0 \leq 4, 0 \leq l_1 < 4, l_0 + l_1 \leq 4\}$.

7. CONCLUSION

In this paper, an intersection theory for generic differential polynomials is presented by giving the explicit formulas for the dimension and order of the intersection of an irreducible differential variety with a generic differential hypersurface. As a consequence, we show that the differential dimension conjecture is true for generic differential polynomials.

The Chow form for an irreducible differential variety is defined. Most of the properties of the algebraic Chow form have been extended to its differential counterpart. In particular, we introduce the concept of differential Chow quasi-variety for a special class of differential algebraic cycles. Furthermore, the generalized Chow form for an irreducible differential variety is also defined and its properties are proved. As an application of the generalized differential Chow form, we can give a rigorous definition for the differential resultant and establish its properties which are similar to that of the Sylvester resultant of two univariate polynomials and the Macaulay resultant of multivariate polynomials.

The results given in this paper enrich the field of differential algebraic geometry. Further, many new problems can be raised naturally. Some of them are already mentioned in Remarks 4.44 and 6.9. We mentioned in Section 1 that the algebraic Chow form has many important applications. It is very interesting to see whether some of these applications can be extended to the differential case.

As we mentioned in Section 5, the theory of differential Chow quasi-varieties is not fully developed, and the main difficulty is to develop an elimination theory for mixed systems with both algebraic and differential equations.

In this paper, we only consider Chow forms for affine differential varieties. It is not difficult to extend most of the results in this paper to the differential Chow form

of differentially projective varieties. Note that differentially projective varieties were defined by Kolchin in [24]. It is expected that Theorems 4.43, 4.45, and 5.7 could be improved for differentially projective varieties.

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